EXPLICIT IMMERSIONS OF A PUNCTURED n-TORUS IN $\mathbb{R}^n$ AND KIRBY'S TORUS TRICK FOR SURFACES AND 3-MANIFOLDS

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Abstract. This is an expository note presenting several low-dimensional applications of Kirby's torus trick. After a discussion of explicit immersions of a punctured n-torus into $\mathbb{R}^n$ that have appeared in the literature [Milnor, Ferry, Barden], we use the torus trick to construct smooth structures on surfaces [Hatcher] and PL structures on 3-manifolds [Hamilton].

Structure of the note The goal of Section 1 is to give an exposition of three explicit constructions of immersions of punctured tori that appeared in the literature. We start by visualizing such an immersion in dimensions 2 and 3 in Section 1.1. Then we look at Milnor's inductive argument in Section 1.2, continue with Ferry's explicit version in Section 1.3 and finish with Barden-Siebenmann's construction Section 1.4 as presented in Rushing's work.

Hatcher's application to smooth structures on surfaces is taken up in Section 2. PL structures on 3-manifolds following Hamilton are treated in Section 3.

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1 Explicit immersions of the n-torus into $\mathbb{R}^n$

An important ingredient for the torus trick, is an immersion of a punctured n-torus into $\mathbb{R}^n$. For this section, we work in the smooth category, where a *smooth immersion* is a smooth map $f\colon M \to N$ between the smooth manifolds M, N such that its differential $T_x(f)\colon T_x(M) \to T_{f(x)}(N)$ is injective at every point $x \in M$. An equivalent condition would be to require that the map f is locally a smooth embedding. Note that an injective immersion is not necessarily a (global) embedding, because an embedding is required to be a homeomorphism onto its image. Wrapping a half-open interval onto a circle, $[0, 1) \to S^1$ is an example of this (an injective immersion which is not a homeomorphism onto its image).

There is a notion of a *topological immersion* between topological manifolds, where the defining property is that every point in the source has a neighborhood on which the map restricts to an embedding.

Remark 1.1. A *smooth submersion* $s\colon M \to N$ between the smooth manifolds M, N is a smooth map $f\colon M \to N$ between the smooth manifolds M, N such that its differential $T_x(f)\colon T_x(M) \to T_{f(x)}(N)$ is surjective at every point $x \in M$. For a map between manifolds of the same dimension, the notions of immersion and submersion coincide. In particular, our immersions $\mathbb{T}^n - \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$ are in this codimension 0 setting. In his letter Milnor uses the 'submersion' terminology, but here we plan to stick with 'immersion'.

The existence of an immersion $\mathbb{T}^n - \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$ can be concluded from Smale-Hirsch theory, which is a tool to study the homotopy type of embedding spaces. In particular, a theorem of Hirsch claims that a smooth, open, parallelizable n-manifold (for example, a punctured n-torus) can be smoothly immersed into $\mathbb{R}^n$.

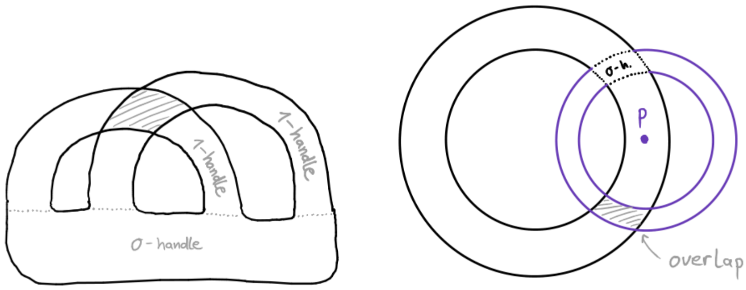


FIGURE 1. Two pictures of an immersion of a punctured 2-torus into the plane.

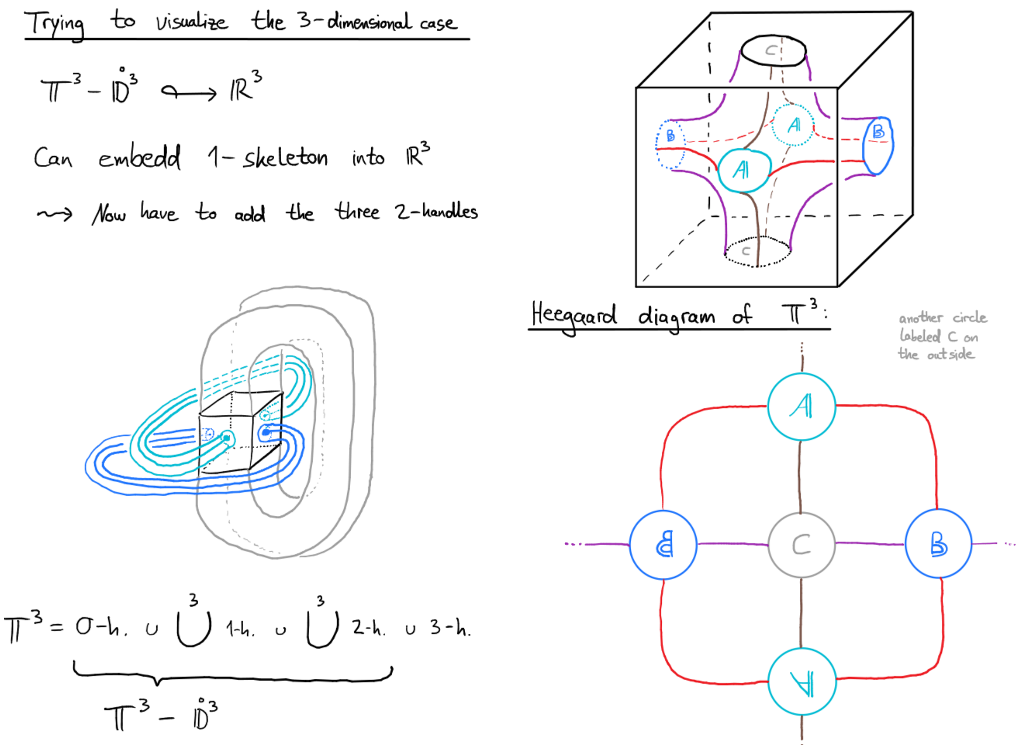


FIGURE 2. Embedding the 1-skeleton of the 3-torus into 3-space.

1.1 Pictures in dimensions 2 and 3

We write $\mathbb{T}^n_0$ for the n-torus where an n-call has been removed. Observe that it does not make a difference whether we remove a point or a closed n-cell.

The 2-torus $\mathbb{T}^2$ has a handle decomposition with one 0-handle, two 1-handles and one 2-handle. An immersion of the 0-handle $D^2 \times D^0$ together with the two 1-handles ($D^1 \times D^1$ attached along $S^0 \times D^1$) into the plane is shown in Figure 1. Since the images of the 1-handles cross, this map is not injective. We can also describe the image of this immersion as the union of two annuli $S^1 \times [0,1] \cup S^1 \times [0,1]$, where one of the overlapping squares takes the role of the 0-handle, while the other square is the region of intersection of the 1-handles.

The 3-torus $\mathbb{T}^3$ has a handle decomposition with one 0-handle, three 1-handles, three 2-handles (which are attached along pairwise commutators of the 1-handles) and a single 3-handle. We would

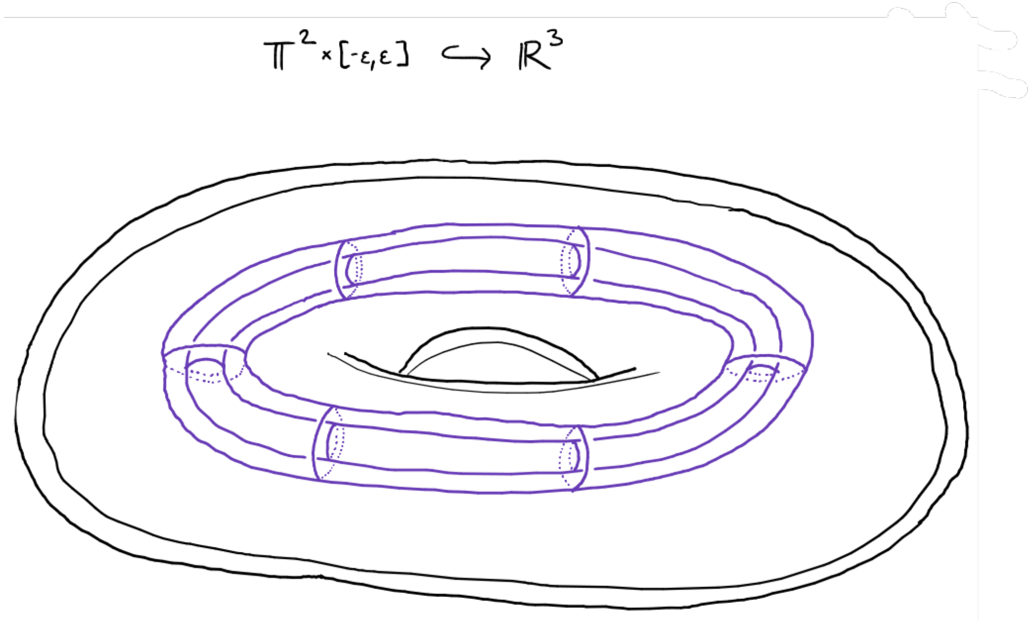


FIGURE 3. Thickened 2-torus with boundary of tubular neighborhood around longitude. This looks like the spin of the immersion in Figure 1.

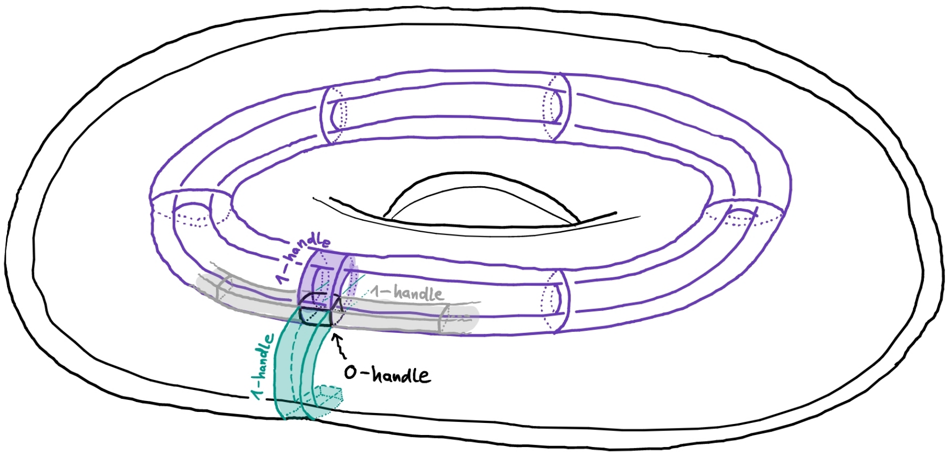


FIGURE 4. Seeing part of the handle decomposition of the 3-torus in the overlaps of the immersion in Figure 3.

like to immerse everything except the top-dimensional handle into 3-space. The 1-skeleton

$$h^0 \cup h^1_a \cup h^1_b \cup h^1_c = D^3 \bigcup_{S^0 \times D^2}^{3} D^1 \times D^2$$

homeomorphic to a 3-dimensional handlebody $\natural^3 S^1 \times D^2$ can be embedded into $\mathbb{R}^3$ as for example in Figure 2. Attaching the 2-handles $D^2 \times D^1$ along $S^1 \times D^1$ so that the attaching spheres $S^1 \times \{0\}$ read off the words $aba^{-1}b^{-1}$, $bcb^{-1}c^{-1}$ and $cac^{-1}a^{-1}$ will introduce (self-)intersections.

The following indication of an immersion $\mathbb{T}^3 - 3\text{-handle} \looparrowright \mathbb{R}^3$ is inspired by Ryan Budney's answer [Bud]. We would like to describe the image of the immersion as the union

$$S^1 \times S^1 \times [0,1] \cup S^1 \times [0,1] \times S^1 \cup [0,1] \times S^1 \times S^1$$

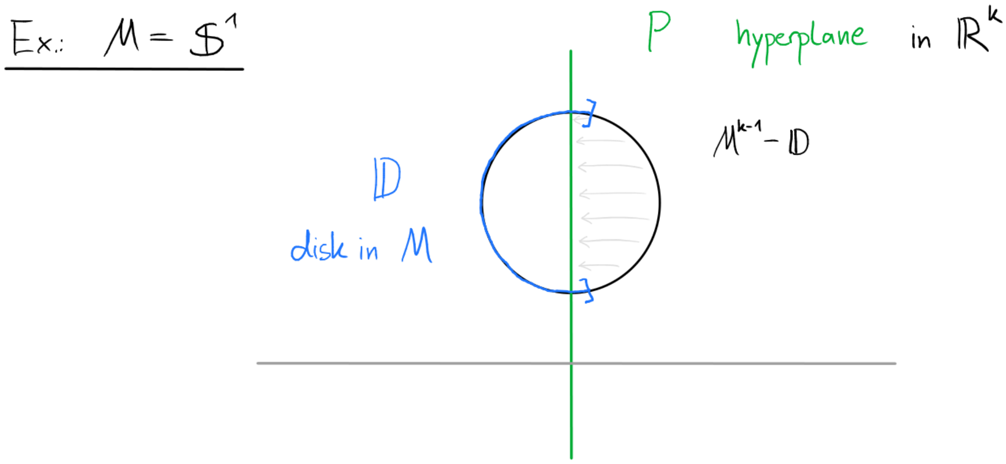


FIGURE 5. The circle S^1 satisfies Milnor's property $\mathfrak{I}$.

of thickened tori (the interval factor $[0,1]$ corresponding to the thickening), where we have to arrange the overlaps so that we can suitably interpret them as handles and double or triple point regions. For example Figure 3, shows an immersion of the union $S^1 \times S^1 \times [0,1] \cup S^1 \times [0,1] \times S^1$ with overlaps compatible to the handle decomposition of the 3-torus, see also Fig. 4. We would still have to add another 2-handle to the picture to complete the immersion of the punctured 3-torus, but we will stop here and move on to the proofs giving general constructions.

1.2 Milnor's inductive argument

Main idea: Milnor's letter printed in [KS77, Essay I, Appendix B]

- Suppose M_k can be embedded in Euclidean space so that projection onto a hyperplane defines an immersion $M_k - \text{disk} \looparrowright \mathbb{R}^k$
- Will show that then also $M_{k+1} = M_k \times S^1$ has this condition
- Starting with $M_1 = S^1$ inductively get immersions of punctured torus

Slogan: Spin and perturb (now can immerse by projecting), or keep going to spin and perturb (and project), ...

Definition 1.2 (Property $\mathfrak{I}$)**.** Let M^{k-1} be a smooth manifold. We say that M *satisfies Property* $\mathfrak{I}$ if it has a codimension 1 embedding into Euclidean space $M^{k-1} \hookrightarrow \mathbb{R}^k$ so that for some smooth closed disk $\mathbb{D} \subset M$ there exists a $k-1$-dimensional hyperplane $P \subset \mathbb{R}^k$ so that the orthogonal projection $\mathrm{pr}_P \colon M - \mathbb{D} \to P$ is an immersion.

Proposition 1.3. *The circle S^1 satisfies property $\mathfrak{I}$.*

Proof. The proof is by picture in Figure 5. □

Theorem 1.4. *If M satisfies Property $\mathfrak{I}$, then so does the product with a circle $M \times S^1$.*

Let us assume the inductive Theorem 1.4 for now, then the immersion of the punctured torus can be built as follows. Inductively, the n-dimensional torus $\mathbb{T}^n = (S^1)^{\times n-1} \times S^1$ satisfies Property $\mathfrak{I}$, so that the orthogonal projection

$$\mathbb{T}^n_0 \cong \mathbb{T}^n - \mathbb{D} \to \mathbb{R}^{n+1} \xrightarrow{\mathrm{pr}_P} P \cong \mathbb{R}^n$$

gives the immersion of the punctured torus.

Proof of Theorem 1.4. Let us make some simplifying assumptions on the embedding of $M^{k-1} \subset \mathbb{R}^k$: We will arrange it so that we can pick the hyperplane $P = \{x_1 = 0\}$ for the immersion of $M - \mathbb{D}$ and that the image of M lies in the open "slab" $\{0 < x_k < \beta\}$ of $\mathbb{R}^k$.

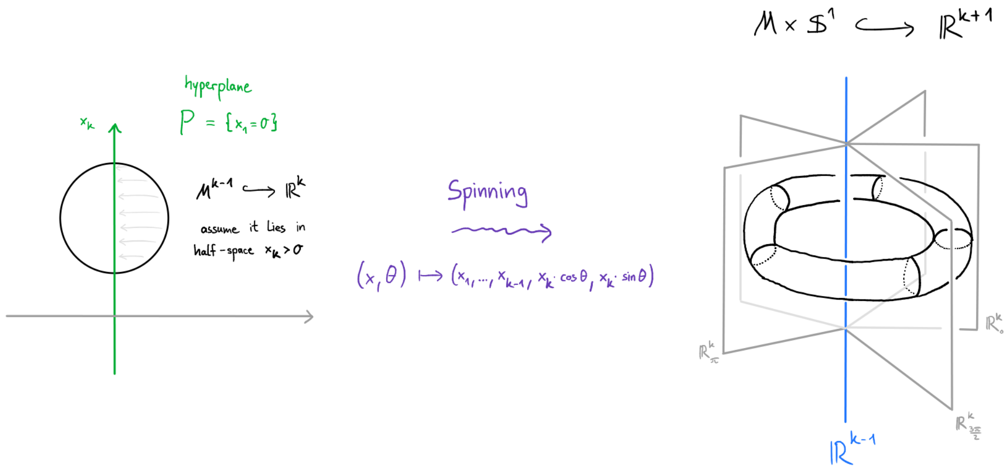


FIGURE 6. Spinning the embedding $M^{k-1} \hookrightarrow \mathbb{R}^k$ (assuming it lies in the half-space $\{x_k > 0\}$) to obtain an embedding $M^{k-1} \times S^1 \hookrightarrow \mathbb{R}^{k+1}$.

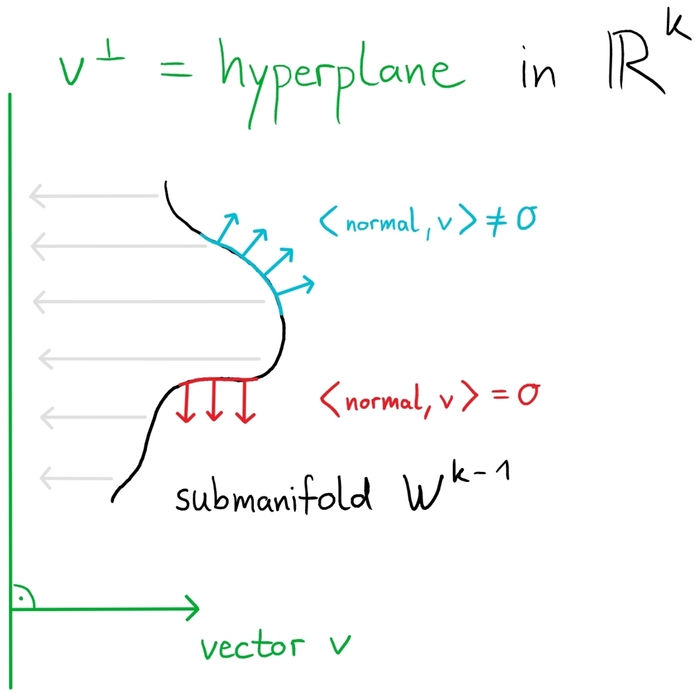


FIGURE 7. Checking that the projection to a hyperplane is an immersion of a submanifold by looking at the normal vector.

Think of $\mathbb{R}^{k+1}$ with its open book decomposition with binding $\mathbb{R}^{k-1}$ and pages the half-spaces $\mathbb{R}^k_+$, as in Figure 6. We can "spin" the subset $M \subset \mathbb{R}^k$ to obtain an embedding

$$M \times S^1 \hookrightarrow \mathbb{R}^{k+1}$$
$$((x_1, \ldots, x_{k-1}, x_k), \theta) \mapsto (x_1, \ldots, x_{k-1}, x_k \cdot \cos\theta, x_k \cdot \sin\theta)$$

Here $\theta \in [0, 2\pi]/0 \sim 2\pi \cong S^1$ is the coordinate on the circle.

We still need a slight deformation of this embedding to check property $\mathfrak{I}$, and find the hyperplane into which we want to immerse.

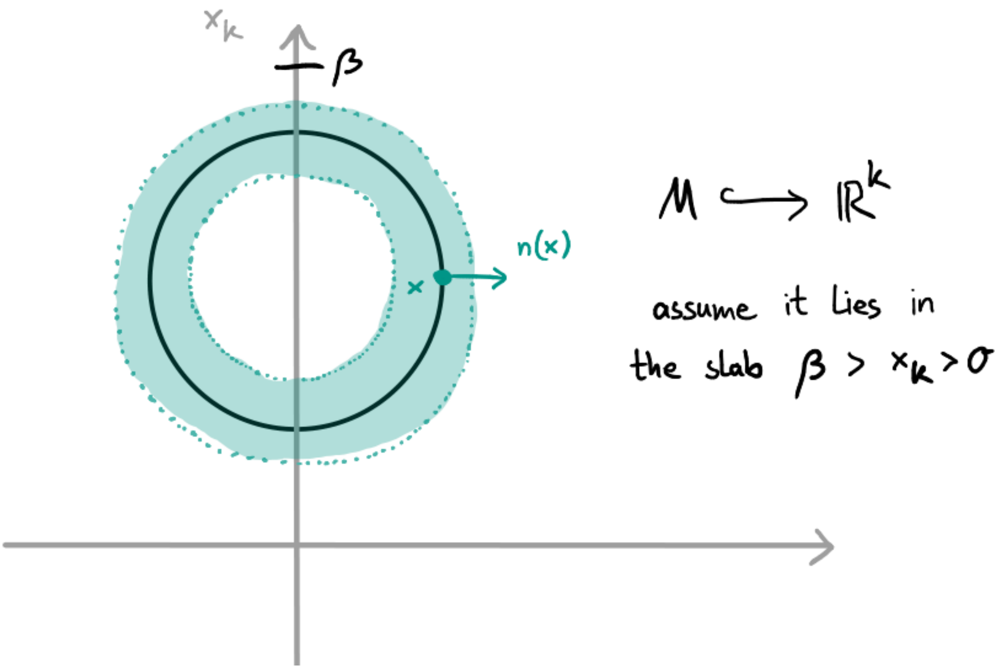


FIGURE 8. Embedding a neighborhood $M^{k-1} \times (-\varepsilon, \varepsilon)$ into $\mathbb{R}^k$.

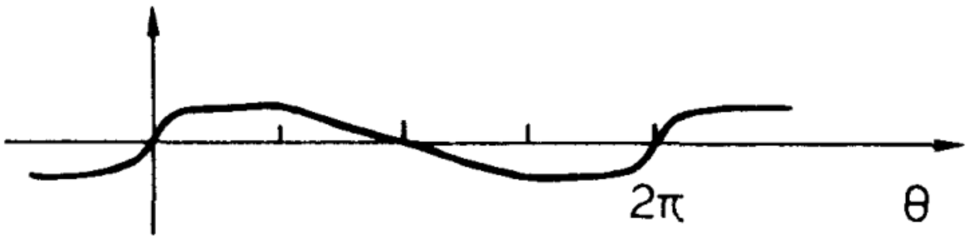


FIGURE 9. An example for the function $\mathrm{t}\colon S^1 \to (-\varepsilon, \varepsilon)$. (© Milnor's letter in [KS77])

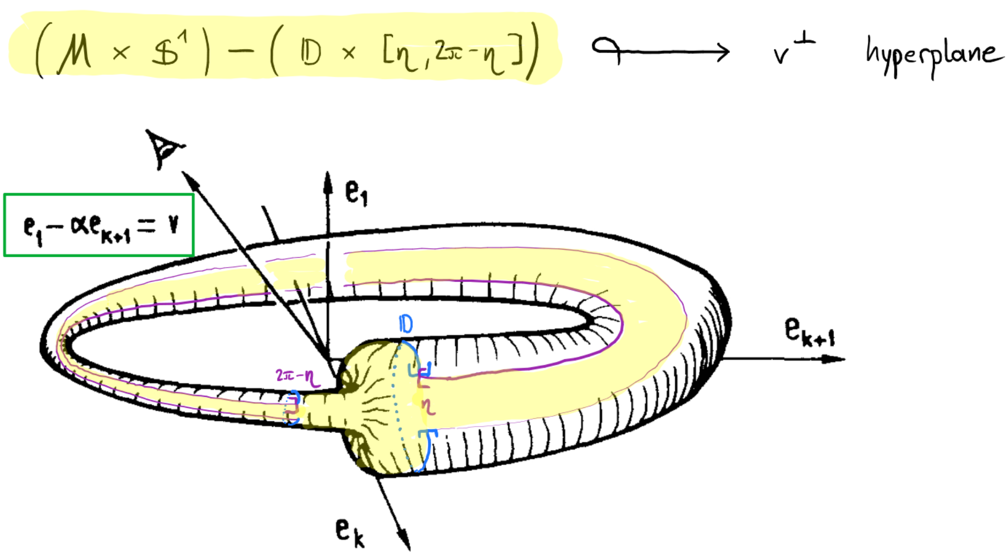


FIGURE 10. Schematic of Milnor's perturbation of the spin. (© Milnor's letter in [KS77])

Let us set up the notation to describe projections to hyperplanes: A normal vector $v \in \mathbb{R}^{k+1}$ determines the hyperplane $v^\perp = \{x \in \mathbb{R}^{k+1} \mid \langle v, x\rangle = 0\}$. Orthogonal projection to $v^\perp$ will give an immersion of a submanifold $W \subset \mathbb{R}^{k+1}$ as long as the normal vector p_w to W at $w \in W$ is not orthogonal to the vector v describing the hyperplane $v^\perp$, see Figure 7. In other words, in our perturbation attempts we want to chose an orthogonal projection direction v so that $\langle p_w, v\rangle > 0$ at all points $w \in W$.

As an Ansatz, look in the direction of the first unit vector $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{k+1}$, and then tilt your head slightly away from the last unit vector $e_{k+1} = (0, \dots, 0, 1) \in \mathbb{R}^{k+1}$. We will try to project to the plane orthogonal to

$$v = e_1 - \alpha e_{k+1}$$

where the amount of tilting $\alpha \in \mathbb{R}_+$ will be determined momentarily.

To parameterize the perturbation of the spin, we use the following equation, whose components will be described in the following enumeration. Also see Figure 10 for a schematic.

$$\begin{aligned} M^{k-1} \times S^1 &\hookrightarrow \mathbb{R}^{k+1} \\ (x, \theta) &\mapsto \mathrm{rot}_\theta(x + \mathrm{t}(\theta) \cdot \mathrm{n}(x)) \end{aligned}$$

(1) The coordinates $x = (x_1, \dots, x_k) \in M$ come from the embedding $M \hookrightarrow \mathbb{R}^k$
(2) $\mathrm{n}(x) = (\mathrm{n}_1(x), \dots, \mathrm{n}_k(x))$ is the unit normal vector to $x \in M^{k-1}$ in $\mathbb{R}^k$.
(3) Since M is compact we can choose an $\varepsilon > 0$ so that (potentially after a translation) the map

$$\begin{aligned} M \times (-\varepsilon, \varepsilon) &\hookrightarrow \mathbb{R}^k \\ (x, t) &\mapsto x + t \cdot \mathrm{n}(x) \end{aligned}$$

is an embedding with image in $\{0 < x_k < \beta\}$, see Figure 8.
(4) The amount by how much we wiggle in the normal direction will vary when we go around the spinning circle, and we specify it with a smooth function $\mathrm{t}\colon S^1 \to (-\varepsilon, \varepsilon), \theta \mapsto \mathrm{t}(\theta)$. We require two further properties of this function t, the graph of an example is shown in Figure 9.
   - $\cos\theta \frac{d\,\mathrm{t}}{d\theta} \geq 0$ for all $\theta \in S^1$.
   - At $\theta = 0$, $\frac{d\,\mathrm{t}}{d\theta} \geq \frac{2\beta}{\alpha}$. Remember that $\beta > 0$ was an upper bound on the x_k-coordinate of the embedding of $M^{k-1} \times (-\varepsilon, \varepsilon)$ into $\mathbb{R}^k$. Now we want to specify $\alpha > 0$ which is also the amount by how much we tilt our projection axis away from the e_{k+1}-direction: 2α is supposed to be a positive lower bound for the first component $\mathrm{n}_1(x)$ of the normal vector for $x \in M - \mathbb{D}$. Such a lower bound exists, since by assumption projecting to $\{x_1 = 0\}$ was an immersion on $M - \mathbb{D}$, so the normal vector cannot be orthogonal to e_1 on (the closure) of this set.

(5) The rotation of the spinning can be encoded in matrix form as

$$\mathrm{rot}_\theta\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$$

$$\mathrm{rot}_\theta = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix}$$

We will now check that projecting to the hyperplane $v^\perp$, $e_1 - \alpha e_{k+1}$, gives an immersion of the perturbed $(M \times S^1) - \text{disk}$. For this, we check the condition $\langle p(x, \theta), v\rangle > 0$, where

$$p(x, \theta) = p((x_1, \dots, x_k), \theta) = (x_k + \mathrm{t}(\theta) \cdot n_k(x)) \cdot \mathrm{rot}_\theta(\mathrm{n}(x)) - \frac{d\,\mathrm{t}(\theta)}{d\theta}$$

is the normal vector to the perturbed embedding of $M \times S^1$. We can compute the scalar product as

$$\langle p(x,\theta), v\rangle = \underbrace{(x_k + \mathrm{t}(\theta)\cdot \mathrm{n}_k(x))\cdot(n_1(x) - \alpha\cdot\sin\theta\cdot\mathrm{n}_k(x))}_{A} + \underbrace{\alpha\cdot\cos\theta\cdot\frac{d\,\mathrm{t}(\theta)}{d\theta}}_{B\geq 0}$$

Bounding this from below splits up into two cases:

- For $x \in M - \mathbb{D}$, arbitrary θ: Remember that α was a lower bound for the first component of the normal vector on this set, so we conclude for the first summand in the scalar product that
$$A \geq (x_k + \mathrm{t}(\theta)\cdot\mathrm{n}_k(x))\cdot(2\alpha - \alpha) > 0$$
The second summand B is non-negative by our construction of the function t. So on this set, $\langle p(x,\theta), v\rangle > 0$
- For all $x \in M$, but $\theta = 0$: Here $A \geq -\beta$ and $B \geq \alpha\frac{2\beta}{\alpha}$. This shows $\langle p(x,0), v\rangle > 0$ and by continuity $\langle p(x,\theta), v\rangle > 0$ for all θ which are sufficiently close to 0, say for $|\theta| \leq \eta$, $\eta > 0$.

In conclusion, projecting to the hyperplane $v^\perp$ is an immersion on

$$(M \times S^1) - (\mathbb{D} \times [\eta, 2\pi - \eta]) \looparrowright v^\perp$$

which is $M \times S^1$ without a disk. This concludes the proof that $M \times S^1$ satisfies property $\mathfrak{I}$. $\square$

Remark 1.5. Milnor cites the paper [Gra74] which contains another explicit construction of an immersion, but unfortunately we were not able to track down this reference.

1.3 Ferry's explicit version

Main idea: [Fer74]

- Define a "standard embedding" $\mathbb{T}^n \times (0,1) \hookrightarrow \mathbb{R}^{n+1}$ via explicit coordinates
- Perturb the image of $\mathbb{T}^n \times \{0\}$ in its normal bundle
- projection to $\mathbb{R}^n$ is an immersion in a neighborhood of the $(n-1)$-skeleton of $\mathbb{T}^n$ (which is a punctured torus)

Slogan: Spin iteratively until we embedded the torus, then after one perturbation at the end we can project to the first coordinates

We will use the coordinates

$$\vec{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n = (S^1)^{\times n}$$

with $\theta_i \in S^1 \cong [0, 2\pi]/0 \sim 2\pi$ to describe points on the n-torus, where we usually pick the representative to lie in the interval $\theta_i \in [0, 2\pi)$.

We will now describe the standard embedding of $\mathbb{T}^n \times (-1,1)$ into $\mathbb{R}^{n+1}$ via an iterated spinning construction. The idea of the spinning is the same as appeared in Milnor's construction, but the difference here is that we will perturb the image only once at the end, and not after each spinning step. The advantage of this is that we can write down the spinning in explicit coordinates.

Start with the standard embedding of the thickened 1-torus

$$\begin{aligned} S^1 \times (-1,1) &\hookrightarrow \mathbb{R}^2 \\ (\theta_1, t) &\mapsto ((1+t)\cdot\cos\theta_1, (1+t)\cdot\sin\theta_1 + 2) \end{aligned}$$

Nos suppose we have constructed an embedding

$$\begin{aligned} \mathbb{T}^n \times (-1,1) &\hookrightarrow \mathbb{R}^{n+1} \\ (\vec{\theta}, t) &\mapsto (f_1(\vec{\theta}, t), \dots, f_n(\vec{\theta}, t), f_{n+1}(\vec{\theta}, t)) \end{aligned}$$

where we assume that we have shifted the last coordinate so that $f_{n+1}(\vec{\theta}, t) > 0$. This assumption on the last coordinate is the reason for the $+2$ in the standard embedding of the 1-torus. Then by spinning we can construct a new embedding

$$\mathbb{T}^{n+1} \times (-1,1) \hookrightarrow \mathbb{R}^{n+2}$$

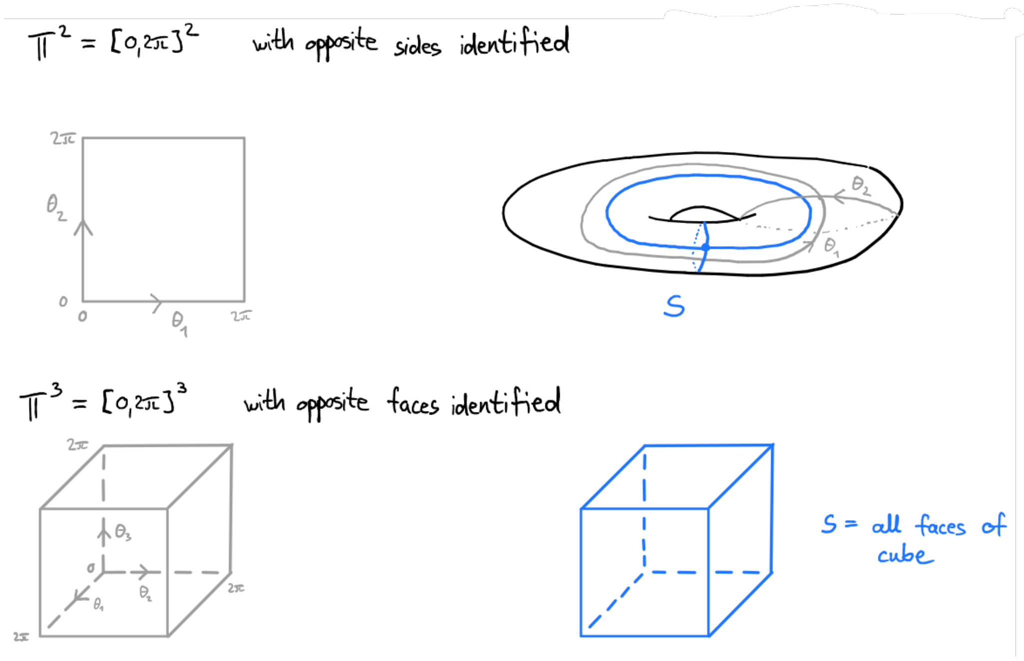


FIGURE 11. The 1-skeleton of the 2-torus $\mathbb{T}^2_{(1)}$ and the 2-skeleton of the 3-torus $\mathbb{T}^3_{(2)}$.

$$(\vec{\theta}, t) \mapsto (f_1(\vec{\theta}, t), \dots, f_n(\vec{\theta}, t), f_{n+1}(\vec{\theta}, t) \cdot \cos\theta_{n+1}, f_{n+1}(\vec{\theta}, t) \cdot \sin\theta_{n+1},)$$

where after each spinning stage, add 2^n to the last coordinate to force the last coordinate to be > 0. Here are the first steps in this construction:

$$\begin{aligned}
\mathbb{T}^1 \times (-1,1) &\hookrightarrow \mathbb{R}^2 \\
(\vec{\theta} = (\theta_1), t) &\mapsto ((1+t) \cdot \cos\theta_1, (1+t) \cdot \sin\theta_1 + 2) \\
\mathbb{T}^2 \times (-1,1) &\hookrightarrow \mathbb{R}^3 \\
(\vec{\theta} = (\theta_1, \theta_2), t) &\mapsto ((1+t) \cdot \cos\theta_1, ((1+t) \cdot \sin\theta_1 + 2) \cdot \cos\theta_2, ((1+t) \cdot \sin\theta_1 + 2) \cdot \sin\theta_2 + 4) \\
\mathbb{T}^3 \times (-1,1) &\hookrightarrow \mathbb{R}^4 \\
(\vec{\theta} = (\theta_1, \theta_2, \theta_3), t) &\mapsto ((1+t) \cdot \cos\theta_1, ((1+t) \cdot \sin\theta_1 + 2) \cdot \cos\theta_2, \\
&\quad (((1+t) \cdot \sin\theta_1 + 2) \cdot \sin\theta_2 + 4) \cdot \cos\theta_3, (((1+t) \cdot \sin\theta_1 + 2) \cdot \sin\theta_2 + 4) \cdot \sin\theta_3 + 8)
\end{aligned}$$

In the circle coordinates, we can explicitly describe the $(n-1)$-skeleton of the n-torus as

$$\mathbb{T}^n_{(n-1)} = \{(\theta_1, \dots, \theta_n) \in \mathbb{T}^n \mid \theta_i = 0 \text{ for some } i \in \{1, \dots, n\}\}$$

Observe that an open neighborhood of $\mathbb{T}^n_{(n-1)} \subset \mathbb{T}^n$ is everything except a closed disk in the n-cell of $\mathbb{T}^n$. See also Figure 11 for an illustration in low dimensions.

Our goal now will be to perturb $\mathbb{T}^n \times \{0\}$ in the normal t-direction so that projecting to the first n coordinates, i.e. to $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, is an immersion on an open tubular neighborhood of $\mathbb{T}^n_{(n-1)}$. See Figure 12 for a schematic illustration. Here the perturbation contains $\varepsilon > 0$ as a small positive parameter, and we pick the function

$$\begin{aligned}
\varphi \colon \mathbb{T}^n &\to \mathbb{R} \\
\vec{\theta} &\mapsto \frac{\sin\theta_1 \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^n} + \frac{\sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^{n-1}} + \ldots + \frac{\sin\theta_{n-1} \cdot \sin\theta_n}{2^2} + \frac{\sin\theta_n}{2}
\end{aligned}$$

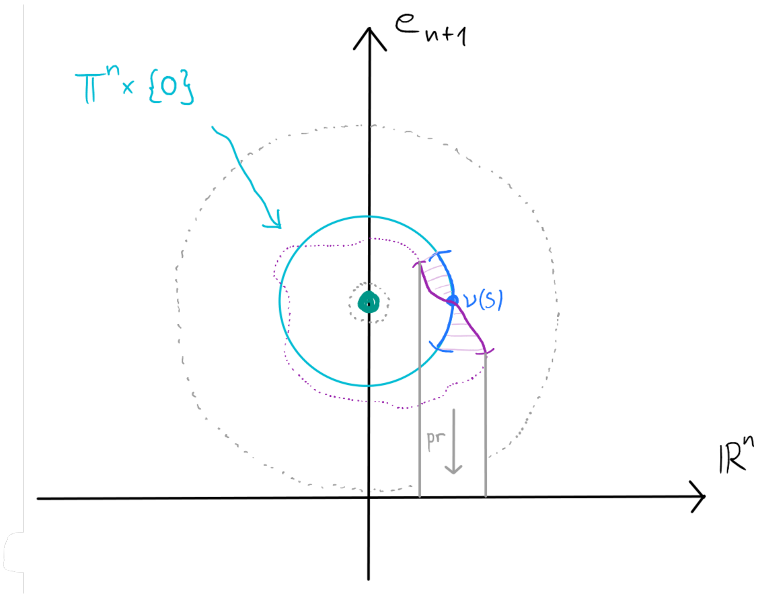


FIGURE 12. Schematic of the projection of the perturbed torus in Ferry's proof

to determine by how much we wiggle in the normal direction. Putting this together, we have the following:

$$\begin{aligned}
\mathbb{T}^n \times (-1,1) &\to \mathbb{R}^{n+1} \\
(\vec{\theta}, t) &\mapsto (f_1(\vec{\theta}, t), \dots, f_n(\vec{\theta}, t), f_{n+1}(\vec{\theta}, t)) \\
\rightsquigarrow \mathrm{pert}\colon \mathbb{T}^n &\to \mathbb{R}^{n+1} \\
\vec{\theta} &\mapsto (f_1(\vec{\theta}, \varepsilon \cdot \varphi(\vec{\theta})), \dots, f_n(\vec{\theta}, \varepsilon \cdot \varphi(\vec{\theta})), f_{n+1}(\vec{\theta}, \varepsilon \cdot \varphi(\vec{\theta}))) \\
\rightsquigarrow \mathrm{pr} \circ \mathrm{pert}\colon \mathbb{T}^n &\to \mathbb{R}^n \\
\vec{\theta} &\mapsto (f_1(\vec{\theta}, \varepsilon \cdot \varphi(\vec{\theta})), \dots, f_n(\vec{\theta}, \varepsilon \cdot \varphi(\vec{\theta})))
\end{aligned}$$

Showing that this composition of the perturbation with the projection has injective differential in a neighborhood of the $(n-1)$-skeleton would prove that is restricts to an immersion of $\mathbb{T}^n - D^n$ into $\mathbb{R}^n$ as desired.

We will skip the calculation, but now it is possible to compute that the differential of the map $\mathrm{pr} \circ \mathrm{pert}$ at points $\vec{\theta} \in \mathbb{T}^n_{(n-1)}$ and $t = \varepsilon \cdot \varphi(\vec{\theta}) = 0$ is given by

$$\frac{-\varepsilon}{2^n} \cdot \det(Df)$$

where Df is the determinant of the Jacobian of the standard embedding of the n-torus into $\mathbb{R}^n$. For details of the computation see Barden's paper [Fer74]. Here we will be content with observing that this Jacobian of the standard embedding is non-singular, and so by continuity of the differential the determinant of $D(\mathrm{pr} \circ \mathrm{pert})$ is non-zero in a small open neighborhood of $\mathbb{T}^n_{(n-1)}$ and for small parameters ε. This concludes our exposition of Ferry's construction.

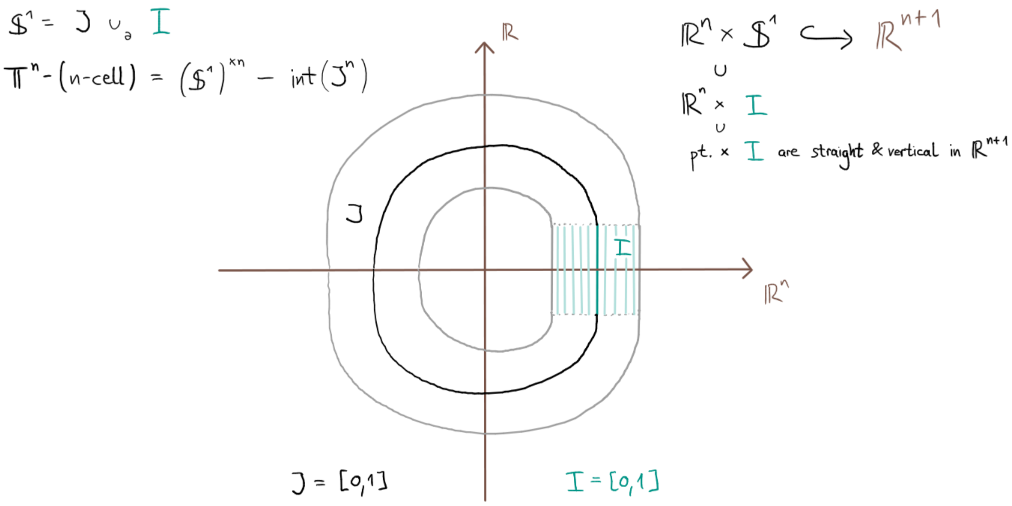


FIGURE 13. Setting up the notation for the subsets of the n-torus, where the circle factors are $S^1 = I \cup_\partial J$. Also pictures is an embedding $\mathbb{R}^n \times S^1 \hookrightarrow \mathbb{R}^{n+1}$. where the I-fibres $\{\text{pt.}\} \times I \subset \mathbb{R}^n \times I \subset \mathbb{R}^n \times S^1$ are straight and vertical in $\mathbb{R}^{n+1}$.

1.4 Barden's inductive proof

Main idea: [Rus73, Immersion Lemma 5.6.1]
- Inductively build immersions $\mathbb{T}^n \times \mathbb{I} \looparrowright \mathbb{R}^n \times \mathbb{I}$
- they restrict to a product map on $(\mathbb{T}^n - n\text{-cell}) \times \mathbb{I}$
- the first factor of the product map gives the desired immersion

Slogan: Add an extra dimension useful for the induction, then restrict to the first factor

This section closely follows Chapter 5 in Rushing's book [Rus73]. The proof originates from Barden, with contributions to the exposition by Edwards and Siebenmann.

We write $\mathbb{T}^n_0$ for the n-torus where an n-cell has been removed.

Proposition 1.6 ((Bard$_n$) = Inductive statement in dimension n)**.** *There exists an immersion*

$$f\colon \mathbb{T}^n \times [-1,1] \looparrowright \mathbb{R}^n \times [-1,1]$$

such that the restriction to $\mathbb{T}^n_0 \times [-1,1]$ is a product map, that is

$$f|_{\mathbb{T}^n_0 \times [-1,1]} = \alpha \times \mathrm{Id}_{[-1,1]} \colon \mathbb{T}^n_0 \times [-1,1] \looparrowright \mathbb{R}^n \times [-1,1]$$

We will prove Proposition 1.6 inductively. Then $\alpha\colon \mathbb{T}^n_0 \looparrowright \mathbb{R}^n$ is the immersion of the punctured torus that we are looking for.

Proposition 1.7 ((Bard$_1$) = Base case)**.** *There exists an immersion*

$$f\colon \mathbb{T}^1 \times [-1,1] \looparrowright \mathbb{R}^1 \times [-1,1]$$

such that the restriction to $\mathbb{T}^1_0 \times [-1,1]$ is a product map, that is

$$f|_{\mathbb{T}^1_0 \times [-1,1]} = \alpha \times \mathrm{Id}_{[-1,1]} \colon \mathbb{T}^1_0 \times [-1,1] \looparrowright \mathbb{R}^1 \times [-1,1]$$

We will use this opportunity to set up some notation for the inductive step, also see Figure 13. We will write the circle $S^1 = I \cup_\partial J$ as the endpoint-union of two interval segments $I = [-1,1] = J$. Then, we can use J^n as the n-cell of the product $\mathbb{T}^n = (S^1)^{\times n}$, and identify $\mathbb{T}^n_0 = \mathbb{T}^n - J^n$. Figure 13 also shows an embedding $\mathbb{R}^n \times S^1 \hookrightarrow \mathbb{R}^{n+1}$ where the I-fibres $\{\text{pt.}\} \times I \subset \mathbb{R}^n \times I \subset \mathbb{R}^n \times S^1$ are straight and vertical in $\mathbb{R}^{n+1}$.

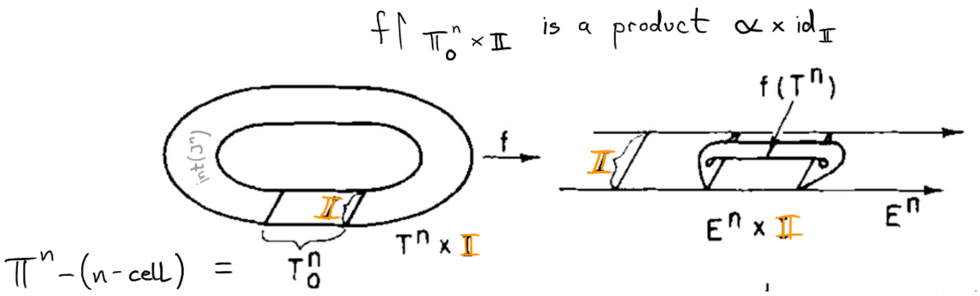


FIGURE 14. Base case: An immersion $f\colon \mathbb{T}^1 \times [-1,1] \looparrowright \mathbb{R}^1 \times [-1,1]$ such that the restriction to $\mathbb{T}^1_0 \times [-1,1] = I \times [-1,1]$ is a product map. (© [Rus73])

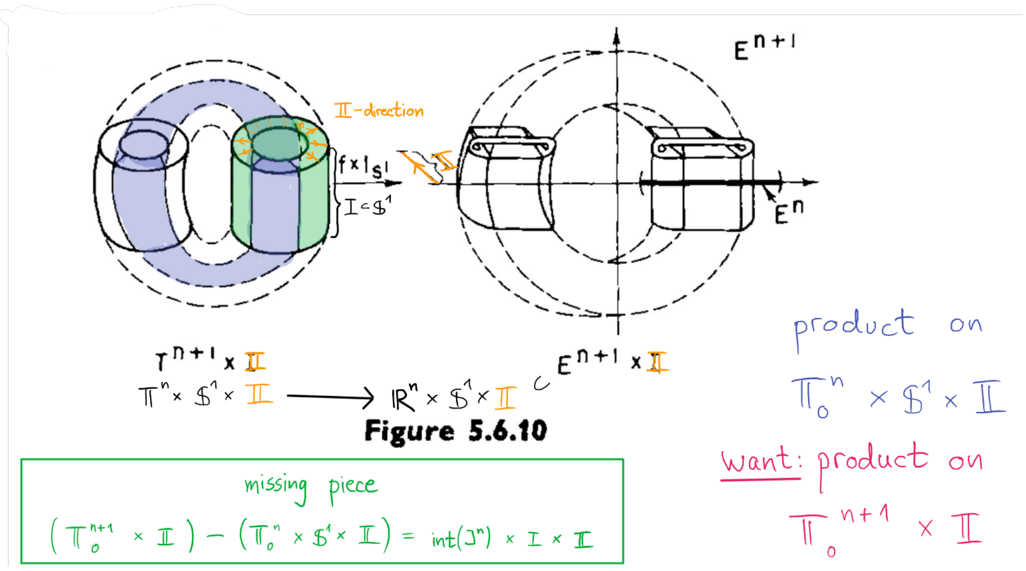


FIGURE 15. Inductive step. (© [Rus73])

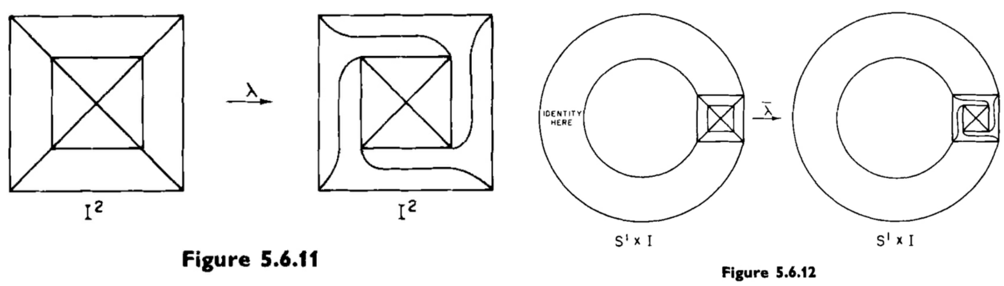


FIGURE 16. The homeomorphism $\lambda\colon [-1,1]^2 \to [-1,1]^2$ which is the identity on the boundary $\partial([-1,1]^2)$, and a $\frac{\pi}{2}$-rotation on the smaller square $[-\frac{2}{3},\frac{2}{3}]^2$. On the right is a picture of extending the map via the identity to a homeomorphism $\overline{\lambda}\colon S^1 \times [-1,1] \to S^1 \times [-1,1]$. (© [Rus73])

Proof of base case (Bard_1) *in Proposition 1.7.* The immersion which is a product on the punctured 1-torus is pictured in Figure 14. □

Proof sketch of the inductive step (Bard_n) $\Rightarrow$ (Bard_{n+1}) *for Proposition 1.6.* Assume $f\colon \mathbb{T}^n \times [-1,1] \looparrowright \mathbb{R}^n \times [-1,1]$ is given so that $f|_{\mathbb{T}^1_0 \times [-1,1]} = \alpha \times \mathrm{Id}_{[-1,1]}\colon \mathbb{T}^1_0 \times [-1,1] \looparrowright \mathbb{R}^1 \times [-1,1]$ is a product map.

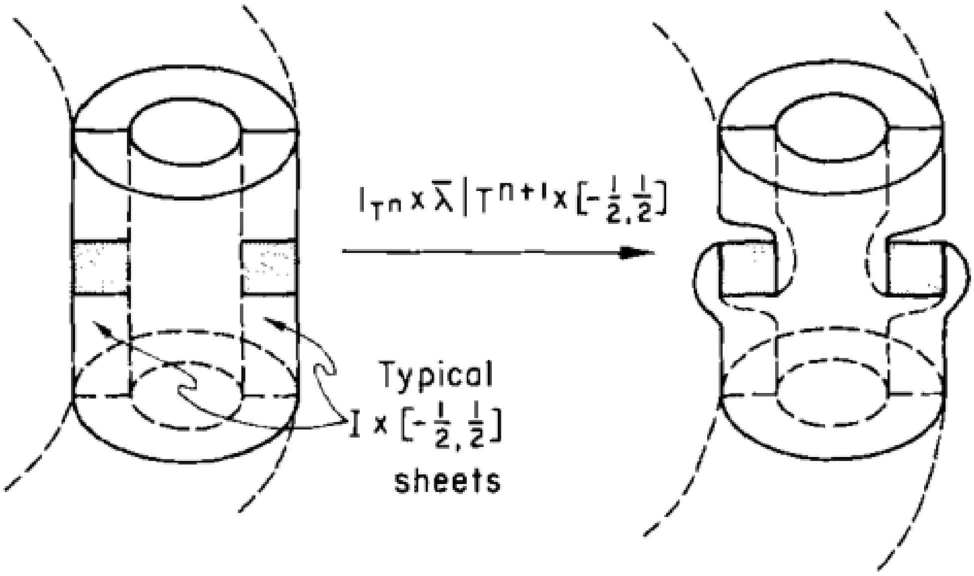


Figure 5.6.13

FIGURE 17. Conjugation with λ applied to several squares. (© [Rus73])

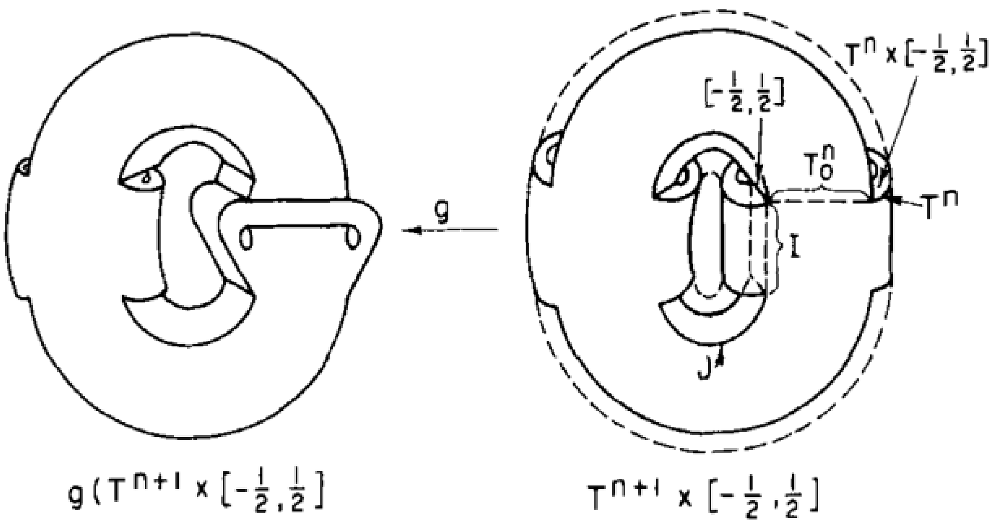


Figure 5.6.14

FIGURE 18. The global situation in the final construction. (© [Rus73])

By crossing with another circle factor and composing with the embedding $\mathbb{R}^n \times S^1 \hookrightarrow \mathbb{R}^{n+1}$ from Figure 13 we can construct an immersion

$$\widetilde{f}\colon \mathbb{T}^n \times S^1 \times [-1,1] \xrightarrow{f \times \mathrm{Id}_{S^1}} \mathbb{R}^n \times S^1 \times [-1,1] \hookrightarrow \mathbb{R}^{n+1} \times [-1,1]$$

For an illustration of the inductive step, see Figure 15. Check that $\widetilde{f}$ is a product on $\mathbb{T}_0^n \times S^1 \times [-1,1]$[1]. We want to construct an immersion which is a product on $\mathbb{T}_0^{n+1} \times [0,1]$, so we have to correct for this on the missing piece

$$(\mathbb{T}_0^{n+1} \times [0,1]) - \mathbb{T}_0^n \times S^1 \times [-1,1] = \operatorname{Int} J^n \times I \times [-1,1]$$

We will do this by conjugating with a 90 degree rotation on the $I \times [-1,1]$ factor, which is possible because $\widetilde{f}|_{\mathbb{T}_0^n \times I \times [-1,1]}$ is a product on the $I \times [-1,1]$ factor.

For convenience, assume that the map $f\colon \mathbb{T}^n \times [-1,1] \looparrowright \mathbb{R}^n \times [-1,1]$ satisfies $f(\mathbb{T}^n \times [-\frac{1}{2}, \frac{1}{2}]) \subset \mathbb{R}^n \times [-\frac{2}{3}, \frac{2}{3}]$. Now see Figure 16 for a description of the "rotation homeomorphism" $\overline{\lambda}\colon S^1 \times [-1,1] \to S^1 \times [-1,1]$ by which we will conjugate. With this setup, we consider the following immersion

$$h\colon \mathbb{T}^n \times S^1 \times [-1,1] \looparrowright \mathbb{R}^{n+1} \times [-1,1]$$
$$h = (\mathrm{Id}_{\mathbb{R}^n} \times \overline{\lambda}^{-1}) \circ \widetilde{f} \circ (\mathrm{Id}_{\mathbb{T}^n} \times \overline{\lambda})$$

The remaining ideas are contained in Figure 17 and Figure 18, see the reference [Rus73] for the concluding arguments. □

2 Torus trick for surfaces

A slogan that is often heard in manifold theory is that 'the categories are the same' in dimension ≤ 3. That is to say there is no difference between smooth, PL, or topological manifolds in these low dimensions. The aim of this section is to elucidate this idea in dimension 2, i.e. for surfaces. This will be achieved via proving the following two theorems, the proofs of which will use the torus trick. The discussion will follow [Hat13].

Theorem 2.1. *Every topological surface can be given a smooth structure.*

Theorem 2.2. *Every homeomorphism of smooth surfaces is isotopic to a diffeomorphism.*

Putting these two theorems together, we get the immediate corollary:

Corollary 2.3. *Every topological surface can be given a smooth structure, which is unique up to diffeomorphism.*

This result is the precise statement hiding behind the slogan 'the categories are the same'. We can also use this result to classify topological surfaces, since it means that the topological classification immediately follows from the smooth classification of surfaces. The proofs of these theorems will use the handle smoothing theorem which we will state and use in Section 2.1.

2.1 Handle smoothing

Here we will state the handle smoothing theorem and use it to prove Theorem 2.1 and Theorem 2.2. We will prove the handle smoothing theorem in Section 2.3.

Theorem 2.4. *Let n and k be non-negative integers such that $n+k=2$ and let $h\colon B^k \times \mathbb{R}^n \to \mathbb{R}^2$ be a topological embedding which is smooth in a neighbourhood of $\partial(B^k \times \mathbb{R}^n)$. Then h may be (topologically) isotoped to a smooth embedding on $B^k \times B^n$, staying fixed near $\partial(B^k \times \mathbb{R}^n)$ and outside a larger neighbourhood of $B^k \times \{0\}$.*

Lemma 2.5. *An open set $W \subset \mathbb{R}^2$ admits a triangulation such that the size of the simplices approaches 0 on the (topological) boundary of W.*

Proof. We prove this by simply constructing such a triangulation. Divide $\mathbb{R}^2$ into unit squares by drawing lines parallel to the x and y-axis.

– Step 1: Throw away all squares that lie entirely outside of W.

[1]This uses the property that the embedding $\mathbb{R}^n \times S^1 \hookrightarrow \mathbb{R}^{n+1}$ has vertical I-fibres.

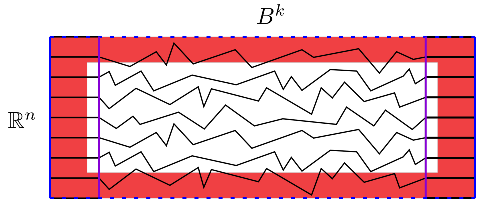


FIGURE 19. Smoothing a handle $B^k \times \mathbb{R}^n$ which is already smooth near $\partial B^k \times \mathbb{R}^n$, staying fixed in the red region.

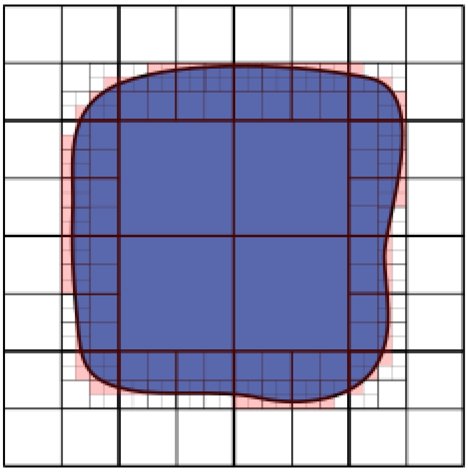


FIGURE 20. Four iterations of steps 1-2 shown for an open set in $\mathbb{R}^2$ (shown in blue). The squares that have not been thrown away are shaded in red.

– Step 2: Divide squares that lie partially inside W into four $\frac{1}{2} \times \frac{1}{2}$ squares each.

Repeat these steps indefinitely (see Fig. 20). The union of the remaining squares is now W and the size of these squares approaches 0 on the (topological) boundary of W. We then turn this into a triangulation by adding a single vertex at the centre of every square and adding in a new edge connecting this central vertex to each other vertex on the square. □

We now prove the existence of smooth structures on surfaces. Note that we always have local smooth structures on surfaces, induced by the standard Euclidean neighbourhoods about points. The difficulty is in piecing together all of these local structures into a single global structure.

Proof of Theorem 2.1. We first consider the closed case. Let S be a closed surface, and $h_i \colon \mathbb{R}^2 \to S$ be (topological) embeddings such that $h_i(\mathbb{R}^2),\ i = 0, 1, 2, \dots$ form an open cover of S. We now proceed via induction, our base case being covered by the existence of local smooth structures. Assume there exists a smooth structure on $U_{n-1} = \bigcup_{i=1}^{n-1} h_i(\mathbb{R}^2)$, and we want to extend this to a smooth structure on $U_n = \bigcup_{i=1}^{n} h_i(\mathbb{R}^2)$. Let $W \colon = h_n^{-1}(U_{n-1})$. Since h_n is continuous and U_{n-1} is open, W is an open set and we can use Lemma 2.5 to construct a triangulation of W with the size of simplices approaching 0 on the (topological) boundary. This triangulation gives us an induced handle decomposition for W, and we can apply the handle smoothing theorem in turn on each handle to smooth h_n on W. This gives us an isotopy h_n^t such that $h_n^0 = h_n|_W$ and h_n^1 is smooth on W and we need to extend this isotopy onto all of $\mathbb{R}^2$. This is possible since the size of the simplices of our triangulation approaches 0 on the (topological) boundary of W, which means that the isotopy approaches the constant isotopy, and thus can be extended onto all of $\mathbb{R}^2$ via the constant isotopy. Now we have extended the smooth structure onto U_n, and this completes the induction.

The case with boundary is similar, but starts with the existence of a collar neighbourhood of the boundary. This collar is of the form $\partial M \times I$, where ∂M is a closed 1-manifold. Since all 1-manifolds

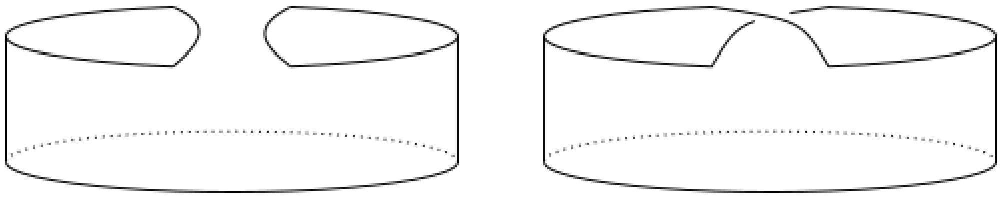


FIGURE 21. Pair of trousers (left) and twisted pair of trousers (right).

are smoothable (see Section 2.4), we know that we can give ∂M a smooth structure and can extend this onto the whole collar. At this point the proof proceeds identically to the closed case, where we start by setting $U_1 \coloneqq \partial M \times I$. □

We now move to proving the uniqueness of smooth structures on surfaces, but first we state and prove another lemma.

Lemma 2.6. *A smooth surface S admits a smooth triangulation.*

By a *smooth triangulation* we mean there exists a simplicial complex S, such that S is homeomorphic to X and the inclusion map $\Delta \to X$ is a smooth embedding for every simplex $\Delta \in S$. We say that a map $\Delta \to X$ is smooth if there exists a smooth extension of the map to an open set $U \supset X$ in $\mathbb{R}^2$.

Proof. The idea of this proof is to construct a smooth cellulation which we then turn into a smooth triangulation. We start by picking a Morse function on our surface S. We can then cut along non-critical levels of our Morse function to cut our surface into smaller pieces. If we only allow a maximum of one critical point to lie between our cuts, then the pieces we can obtain are as follows: if no critical point lies between our cuts, we obtain an annulus; if one index 0 or 2 critical point lies between our cuts, we obtain a disc; if one index 1 critical point lies between our cuts, we obtain either a pair of trousers or a twisted pair of trousers, depending on whether the 1-handle was twisted when attached (see Figure 21). A twisted pair of trousers can be thought of as a punctured Möbius band, and so we can further cut a twisted pair of trousers into a regular pair of trousers and a Möbius band by cutting along a circle that winds twice around the band, avoiding the puncture (see Figure 22).

We now have a decomposition on S into discs, annuli, pairs of trousers, and Möbius bands. This can be turned into a smooth cellulation by adding in one vertex to each boundary circle on every piece, and then adding in edges depending on the type of piece. For discs, we add no edges; for annuli, we add a single edge connected the two vertices directly; for pairs of trousers, we add in two edges connecting two of the boundary circles to the third; for a Möbius band, we add in a single edge connecting the sole vertex to itself, winding all the way along the band. This cuts all of our pieces into polygons, giving us a smooth cellulation. We can then further cut these polygons into triangles by adding an extra vertex in the interior of each piece and connecting it to all other vertices by edges (this step isn't necessary for the Möbius band, which has already been cut into a triangle). This gives us the required smooth triangulation of S. □

Proof of Theorem 2.2. Let $f\colon S \to S'$ be a homeomorphism of smooth surfaces. We want to show that f is isotopic to a diffeomorphism. We start by considering the closed case $\partial S = \emptyset$. Lemma 2.6 gives us a smooth triangulation of S. We can then apply Theorem 2.4 successively. First, we smooth f near the vertices of our triangulation. Every vertex in S has a B^2 neighbourhood which f (topologically) embeds inside a copy of $\mathbb{R}^2 \subset S'$ and hence we can use the Theorem 2.4 to smooth f on this neighbourhood. Next we smooth f near the edges of our triangulation in the analogous manner. Since f is already smooth near the vertices at the ends of each edge, we can isotop f to be smooth on a $B^1 \times B^1$ neighbourhood of the edge and the isotopy stays fixed near the vertices, hence keeping the smoothness of f that we have already achieved. The final step is to smooth f on the faces of our triangulation, and again we can do this precisely because we have already smoothed f near all of the edges and vertices of our triangulation. f is now locally a smooth embedding, and hence a local diffeomorphism.

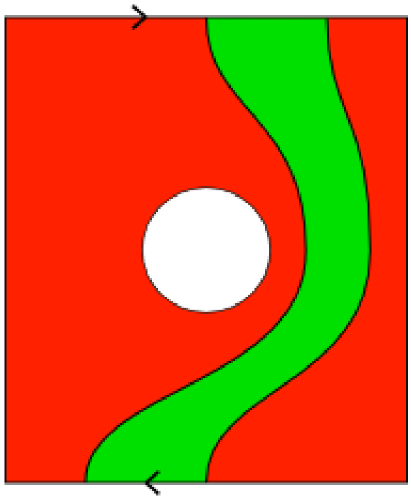


FIGURE 22. Twisted pair of trousers cut along a circle to give a Möbius band (green) and a normal pair of trousers (red).

f is also still injective by its construction and so is a global (topological) embedding of homeomorphic surfaces, and hence must be surjective. Therefore we have isotoped f to a global diffeomorphism.

Now assume ∂S is non-empty. Pick a smooth collar for S, and then glue on another smooth collar $\partial S \times I$, extending f onto it via the identity. We now have a smooth collar such that f is constant with respect to the collar parameter on a smaller sub-collar. Now f restricted to ∂S is a homeomorphism of smooth 1-manifolds and hence is isotopic to a diffeomorphism (see Section 2.4). We can then extend this isotopy onto the subcollar such that it is constant on the internal boundary of the subcollar, allowing us to extend the isotopy onto the rest of S as the constant isotopy. We now have that f is already smooth on a collar of S, and we can then proceed with exactly the same method for the empty boundary case to smooth f on the rest of S, provided that we ensure our smooth triangulation of S restricts to a smooth triangulation of the collar. □

2.2 Studying surfaces using graphs

To prove the handle smoothing theorem, we will need to employ a number of techniques for dealing with smooth surfaces. In this section we will describe the general scheme in which this will be done, which develops the ideas used in the proof of Lemma 2.6.

Let S be a smooth surface, possibly with boundary and choose a Morse function f for S. As in the proof of Lemma 2.6, we cut along non-critical levels of f to obtain pieces P_i, which are discs, annuli, pairs of trousers or Möbius bands. Note that if we allow for non-compact surfaces, then we can get more types of pieces: open-discs, half-open discs ($D^1 \times \mathbb{R}$) etc., but the general idea is the same. We now have a decomposition of our surface into pieces P_i, which are joined together by circles which we will denote by C_j.

We now construct a graph from our surface. Let Γ_S be the graph such that Γ_S has one vertex for every piece P_i and two vertices are connected by an edge for each boundary circle C_j that they share. We then have a natural map $p\colon S \to \Gamma_S$ that maps product neighbourhoods of C_j to their corresponding edges and collapses the remaining portions of the P_j to their corresponding vertices (see Figure 23). Consider the induced map on fundamental groups $p_*\colon \pi_1(S) \to \pi_1(\Gamma_S)$. Since the pieces P_i are path-connected, we can construct well-defined loops in S (up to homotopy) mapping to any loops in Γ_S, so this map must be split surjective. Note that when choosing the segment of the loop in each piece P_i, if the piece is not simply-connected the segment should be chosen such that it is trivial in $\pi_1(P_i)$. Hence, we can conclude that there exists of subgroup of $\pi_1(S)$ which is isomorphic to $\pi_1(\Gamma_S)$.

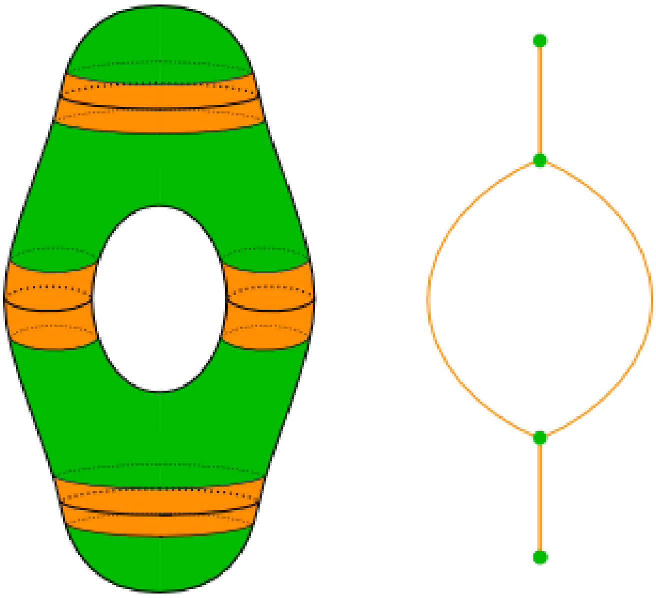


FIGURE 23. Constructing a graph from a torus, cut along four circles into four pieces.

The strength of this viewpoint is that we can simplify our graphs homotopically and have the simplifications pull back to simplifications of our surface. If we have an index-1 vertex on Γ_S, we can remove it and its corresponding edge, leaving a homotopy equivalent graph. Now we see how this change can be pulled back to S. At each C_j, the pieces are glued together via a diffeomorphism of S^1, which up to isotopy are either the identity or the inverse map $z \mapsto z^{-1} = z^*$. When one of the pieces corresponds to an index-1 vertex, this piece must be a disc and this diffeomorphism makes no difference to the diffeomorphism type of the resulting surface. This means that we can alter our Morse function to remove this disc piece, provided that the other piece was not also a disc. If the disc was attached to an annulus, we simply decrease the level at which the index-2 critical point occurs, whereas if the disc was attached to a pair of trousers, we cancel out the index-2 critical point with the index-1 critical point in the pair of trousers. The upshot of this is that we can always simplify finite sub-trees in Γ_S, with the result representing a diffeomorphic surface to S. We illustrate this technique, and end this subsection, with an example.

Example 2.7. Consider a topological torus with some smooth structure $\mathcal{S}$, denoted $T_{\mathcal{S}}$. We want to show that $T_{\mathcal{S}}$ is diffeomorphic to the standard torus T, and we will do this using the graph $\Gamma_{T_{\mathcal{S}}}$. Since $\pi_1(T_{\mathcal{S}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is abelian, and $\pi_1(\Gamma_{T_{\mathcal{S}}})$ is a free subgroup, we know that $\pi_1(\Gamma_{T_{\mathcal{S}}})$ is isomorphic to either $\mathbb{Z}$ or the zero group. If it is the zero group, then $\Gamma_{T_{\mathcal{S}}}$ is a tree and we can cancel sub-trees until we end up with the graph with two vertices connected by a single edge. This must correspond to $T_{\mathcal{S}}$ being diffeomorphic to a sphere, which cannot be true as the fundamental group of $T_{\mathcal{S}}$ is non-trivial. So, the group must be $\mathbb{Z}$, which implies that $\Gamma_{T_{\mathcal{S}}}$ is a circle with finitely many sub-trees attached. Again, we can cancel these sub-trees to obtain a circular graph, which corresponds to $T_{\mathcal{S}}$ being made of finitely many annuli glued together in a circle, corresponding to either a standard Klein bottle or the standard torus (depending on the type of the glueing diffeomorphisms on the C_j). Since $\pi_1(T_{\mathcal{S}})$ does not match that of a Klein bottle, we must conclude that $T_{\mathcal{S}}$ is diffeomorphic to the standard torus.

2.3 Proof of the handle smoothing theorem

We will now prove Theorem 2.4, using the techniques we have just developed along with the torus trick. We will take the cases $k = 0, 1, 2$ separately, as their proofs are very different.

Proof of Theorem 2.4.
$k = 0$ **case, or** 0**-handle smoothing:** It may be useful to refer to Fig. 24 throughout this proof to

visualise the sequence of steps. We begin at the bottom of the diagram, and work our way up to the top. Let $h\colon B^0 \times \mathbb{R}^2 = \mathbb{R}^2 \to S$ be the embedding that we wish to smooth and suppose we are given a fixed (topological) immersion $T^2 \setminus * \looparrowright \mathbb{R}^2$. Such immersions were explicitly constructed in the previous section. We can pull back the smooth structure on S to give a smooth manifold structure on $T^2 \setminus *$ that we will denote as $(T^2 \setminus *)_{\mathcal{S}}$. We want to be able to extend this to a smooth structure on the whole torus, but to do so we need to prove that it is standard near the puncture.

First, we create the graph Γ for $(T^2 \setminus *)_{\mathcal{S}}$ as in Section 2.2. Since $\pi_1((T^2 \setminus *)_{\mathcal{S}})$ is finitely generated, $\pi_1(\Gamma)$ must be also, which means that there exists a finite subgraph Γ_0 such that the closure of $\Gamma \setminus \Gamma_0$ is a disjoint union of finitely many trees. The key here is that since $(T^2 \setminus *)_{\mathcal{S}}$ has only one end, one and only one of these trees must be infinite. Simplify the graph by removing the finite trees and simplify the infinite tree by removing any finite subtrees. These simplifications are simultaneously realised on the surface, which means that there exists a compact set whose complement is diffeomorphic to $S^1 \times \mathbb{R}$, i.e. an infinite number of annuli glued together. This proves that the smooth structure was standard near the puncture, and hence we can extend our smooth structure onto T^2 to give $T^2_{\mathcal{S}}$.

From Example 2.7 we know that all smooth structures on a torus are diffeomorphic, so there exists a diffeomorphism $g\colon T^2_{\mathcal{S}} \to T^2$. We want to lift this diffeomorphism up to a diffeomorphism $\tilde{g}\colon \mathbb{R}^2_{\mathcal{S}} \to \mathbb{R}^2$ of the universal covers, but we first need to normalise g so that it induces the identity map on the fundamental groups. Firstly, we may assume that g maps the basepoint to the basepoint, by rotating the S^1 factors in either the domain or the codomain. Then, note that g being a diffeomorphism implies that the induced map on fundamental groups $\pi_1(g)$ is an isomorphism. $\pi_1(g)^{-1} \in GL_2(\mathbb{Z})$ corresponds naturally to diffeomorphism on T^2 given by the action of $GL_2(\mathbb{Z})$ on $T^2 = R^2/\mathbb{Z}^2$. Post-composing g with this diffeomorphism allows us to assume that g induces the identity map on fundamental groups.

We now have lifted our diffeomorphism g to a diffeomorphism $\tilde{g}\colon \mathbb{R}^2_{\mathcal{S}} \to \mathbb{R}^2$. We would like to extend this to a homeomorphism $G\colon B^2 \to B^2$ that is the identity on the boundary. One way to prove that this is possible is to show that $\tilde{g}$ is bounded, i.e. to show that the set $\{|\tilde{g}(x) - x| \mid x \in \mathbb{R}^2\}$ is bounded above. But this is easy, since we know that $\tilde{g}$ is bounded on $[0,1] \times [0,1]$ by compactness, and thus is bounded on $\mathbb{R}^2$ by periodicity.

If we consider B^2 as the unit disc in R^2, we can then extend G onto R^2 by extending via the identity to construct a map $\tilde{G}\colon \mathbb{R}^2 \to \mathbb{R}^2$. By the Alexander trick, we know this is (topologically) isotopic to the identity, so there exists an isotopy $\widetilde{G}_t$ where $\widetilde{G}_1 = \widetilde{G}$ and $\widetilde{G}_0 = Id$. We now claim that $h_t = G_t^{-1} \circ h$ is the required isotopy that we wanted to construct originally. Clearly $h_0 = h$, so it suffices to show that h_1 is smooth near 0 and that $h_t = h$ far away from 0. Since $\widetilde{G}_t$ is the identity outside of B^2, this second condition is obviously satisfied. To see why the first is satisfied, note that $\widetilde{G}_1$ is a diffeomorphism from the smooth structure $\mathcal{S}$ to the standard smooth structure near 0, and that h is (by definition) smooth on the $\mathcal{S}$ smooth structure. This implies that h_1 is smooth near 0, completing the proof.

$k = 1$ **case, or 1-handle smoothing:** Let $h\colon B^1 \times \mathbb{R} \to S$ be a topological embedding that is already smooth near $\partial B^1 \times \mathbb{R}$. We want to smooth this embedding near $B^1 \times \{0\}$ with an isotopy that stays fixed near $\partial B^1 \times \mathbb{R}$ and outside some larger neighbourhood of $B^1 \times \{0\}$. We can pull the smooth structure on S back to $B^1 \times \mathbb{R}$ to give it a smooth structure which is standard near the boundary. Denote this smooth manifold by $(B^1 \times \mathbb{R})\mathcal{S}$.

We now construct a diffeomorphism $f\colon (B^1 \times \mathbb{R})_{\mathcal{S}} \to B^1 \times \mathbb{R}$. Consider the projection $\pi\colon B^1 \times \mathbb{R} \to \mathbb{R}$. We can perturb this to a Morse function h on $(B^1 \times \mathbb{R})_{\mathcal{S}}$ with $h = \pi$ near $\partial B^1 \times \mathbb{R}$, since π was already smooth there. Note that all of the critical points of h lie in the interior of the smooth manifold by construction. We then construct the graph Γ as in Section 2.2. Since $\pi_1(B^1 \times \mathbb{R}) = 0$, our graph must be a tree. Since $B^1 \times \mathbb{R}$ has two ends, this must be a infinite tree in two directions, with finite sub-trees attached. We can cancel out these finite sub-trees to leave Γ being homeomorphic to $\mathbb{R}$. Our Morse function h is correspondingly altered (staying fixed near $\partial B^1 \times \mathbb{R}$, to remove all critical points. We then use the flow lines of h to construct our required diffeomorphism. Any point $x \in B^1 \times \mathbb{R}$ lies on a unique flow line l_x which passes throught a point p_x on $B^1 \times \{0\}$. We then define $f(x) \colon = (p_x, h(x)$, which is the identity near $\partial B^1 \times \mathbb{R}$ since the flow lines of h are standard there.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\widetilde{G}"] & \mathbb{R}^2 \\
B^2 \arrow[r, "G"] \arrow[u] & B^2 \arrow[u] \\
\mathbb{R}^2_{\mathcal{S}} \arrow[r, "\widetilde{g}"] \arrow[u] \arrow[d] & \mathbb{R}^2 \arrow[u] \arrow[d] \\
T^2_{\mathcal{S}} \arrow[r, "g"] & T^2 \\
(T^2 \setminus *)_{\mathcal{S}} \arrow[u] \arrow[d, hook] & \\
\mathbb{R}^2 \arrow[r, hook, "h"] & S
\end{tikzcd}$$

FIGURE 24. Torus trick diagram for the 0-handle case

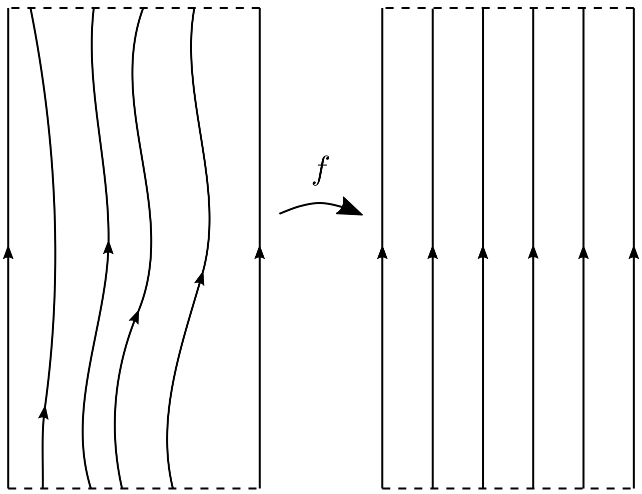


FIGURE 25. The diffeomorphism $f\colon (B^1 \times \mathbb{R})_{\mathcal{S}} \to B^1 \times \mathbb{R}$ fixing a neighbourhood of $\partial B^1 \times \mathbb{R}$ and sending flow lines of h to flow lines of π.

This means we can extend our smooth structure onto $B^1 \times S^1$ such that there exists a diffeomorphism $g\colon (B^1 \times S^1)_{\mathcal{S}} \to B^1 \times S^1$ which is the identity near $\partial B^1 \times S^1$. We then proceed in a similar manner to the $k = 0$ case by normalising g such that it induces the identity on fundamental groups and then lifting to a diffeomorphism $\tilde{g}\colon (B^1 \times \mathbb{R})\mathcal{S} \to B^1 \times \mathbb{R}$. By the same argument for the $k = 0$ case, $\tilde{g}$ must be bounded and hence we can radially reparameterise on the second factor and extend by the identity to receive a map $G\colon B^1 \times B^1 \to B^1 \times B^1$ which is the identity near $\partial B^1 \times B^1$ and matches g near $B^1 \times \{0\}$.

The final step is to then extend G by the identity to a diffeomorphism $\widetilde{G}\colon B^1 \times \mathbb{R} \to B^1 \times \mathbb{R}$. Then apply the Alexander trick to $B^1 \times B^1$ to construct an isotopy $\widetilde{G}_t$ from $\widetilde{G}_1 = \widetilde{G}$ to $\widetilde{G}_0 = \mathrm{Id}$. Since $\widetilde{G}$ is already the identity outside of $B^1 \times B^1$ and near $\partial B^1 \times \mathbb{R}$, we may assume the isotopy fixes both of these regions. Thus, $h_t = h \circ \widetilde{G}_t^{-1}$ is the desired smoothing isotopy, completing the proof.

$k = 2$ **case, or** 2**-handle smoothing:** Let $h\colon B^2 \to S$ be a topological embedding that is already smooth near ∂B^2. We want to smooth this embedding completely with an isotopy that stays fixed near ∂B^2. First, pull the smooth structure on S back onto B^2 to form $B^2_{\mathcal{S}}$ which has the standard

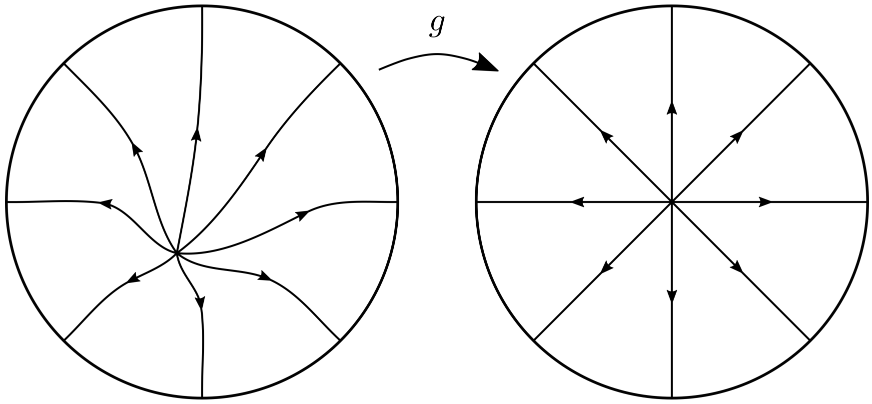


FIGURE 26. The diffeomorphism $g\colon B^2_{\mathcal{S}} \to B^2$ fixing a neighbourhood of ∂B^2 and sending flow lines for $\tilde{r}$ to flow lines of r.

smooth structure near ∂B^2. We do not have any form of torus trick available to us so we will have to construct a diffeomorphism that satisfies our requirements on our own.

Let $r\colon B^2 \to [0,1]$ be the radial function on B^2. If we consider the restriction of r to a neighbourhood of ∂B^2 we can extend this to a Morse function $\tilde{r}\colon B^2_{\mathcal{S}} \to [0,1]$ since the smooth structure is standard near ∂B^2. Since we understand the behaviour of $\tilde{r}$ near the boundary, we know that all the critical points of $\tilde{r}$ must lie in the interior of the disc. We can then construct Γ for $B^2_{\mathcal{S}}$ as before. Since $\pi_1(B^2) = 0$, we know that Γ is a tree and hence we can simplify it down to a single point. This means that $\tilde{r}$ can be simplified to have only a single critical point of index 0. We then construct a diffeomorphism $g\colon B^2_{\mathcal{S}} \to B^2$. Every point x in $B^2_{\mathcal{S}}$ aside from the critical point of $\tilde{r}$ lies on a unique flow line l_{p_x} ending at a point $p_x \in \partial B^2$ and g maps x to $g(x)$ where $g(x)$ lies on the unique flow line ending at the point p_x for the radial function on B^2 such that $r(g(x)) = \tilde{r}(x)$. Finally, the critical point of $\tilde{r}$ is mapped to $0 \in B^2$. By construction, this map must a diffeomorphism that fixes a neighbourhood of the boundary.

Now the Alexander trick gives us an isotopy G_t of g to the identity which we may assume to be fixed near ∂B^2, i.e. $G_0 = \mathrm{Id}$, $G_1 = g$. Our required isotopy is then given by $h \circ G_t^{-1}$. This finishes the $k = 2$ case and hence finishes the whole proof. □

2.4 Smoothing and classifying one-dimensional topological manifolds

In our proofs of Theorem 2.1 and Theorem 2.2 we used that analogous results hold for 1-manifolds. Here we give the outline of how to prove these results. It is much easier than the surfaces case and so the treatment will be less detailed (so as to not labour the point). We will discuss how to prove a 1-dimensional handle smoothing theorem, leaving it to the reader to apply it to obtain existence and uniqueness of smooth structures for topological 1-manifolds. We will use the smooth classification of 1-manifolds to do this (for a proof of this, see [Mil97, appendix]).

0**-handle smoothing:** Let $h\colon \mathbb{R} \hookrightarrow \mathcal{O}$ be a topological embedding into a smooth 1-manifold $\mathcal{O}$. We can pull the smooth structure on $\mathcal{O}$ back onto $\mathbb{R}$. Now consider a topological 'immersion' $S^1 \setminus * \looparrowright \mathbb{R}$, which must in fact be a topological embedding of an open interval. We can then pull the smooth structure induced by h onto this open interval to form $(S^1 \setminus *)_{\mathcal{O}}$, which by the classification of smooth 1-manifolds must be diffeomorphic to the standard interval. Hence we can extend this smooth structure onto the circle to form a smooth manifold $S^1_{\mathcal{O}}$. Again by the classification of smooth 1-manifolds, there exists a diffeomorphism $f\colon S^1_{\mathcal{O}} \to S^1$. We then normalise f so that it maps $1 \in S^1$ to itself, and since f already must induce the identity homomorphism on π_1, this means that f lifts to a map on the universal covers $\widetilde{f}\colon \mathbb{R}_{\mathcal{O}} \to \mathbb{R}$.

It is not hard to see now, following the proof of 0-handle smoothing for surfaces, how we can construct a diffeomorphism $\widetilde{F}\colon \mathbb{R} \to \mathbb{R}$ isotopic to the identity, such that $\widetilde{F}$ is the identity outside of D^1 and $h \circ \widetilde{F}^{-1}$ is a smooth embedding.

1-handle smoothing: Let $h\colon I \hookrightarrow \circledcirc$ be a topological embedding that is smooth near ∂I. We can pull the smooth structure on $\circledcirc$ back onto I, to form a smooth manifold $I_{\circledcirc}$ which will have the standard structure near ∂I. We can then decompose $I_{\circledcirc}$ as $I \cup \widetilde{I} \cup I$, two standard smooth intervals glued to either end of a possibly non-standard interval. But by the classification of smooth 1-manifolds, $\widetilde{I}$ is diffeomorphic to the standard interval, and so, possibly after smoothing glueing points, we have a diffeomorphism $f\colon I_{\circledcirc} \to I$ which is the identity near the boundary. By the Alexander trick, f is topologically isotopic to the identity, and this isotopy gives the required smoothing.

Using this handle smoothing to obtain existence and uniqueness results for smooth structures on topological 1-manifolds, this allows us to now classify topological 1-manifolds. Since the smooth classification of 1-manifolds says that there are only four such manifolds: the circle, the open interval, the closed interval and the half-open interval, these must also be the only topological 1-manifolds.

3 Torus trick for 3-manifolds

In this section we present a version of the torus trick for 3-manifolds due to Hamilton [Ham76]. In particular, we will describe an alternative proof of the theorem that every topological 3-manifold admits a unique PL structure up to isotopy using the torus trick. As we will see, this follows from a 3-dimensional version of the handle straightening theorem. By default, we assume that a manifold is second-countable.

3.1 The 3-dimensional handle straightening theorem

Recall that in lectures we discussed a CAT handle straightening theorem where CAT is PL or DIFF for manifolds of dimension 5 or higher (see Theorem 19.1 in the lecture notes). In Section 2 we describe a similar result for surfaces for CAT=DIFF. In this section, we prove a PL-handle straightening theorem for 3-manifolds.

We call an PL n-manifold *irreducible* if every PL $(n-1)$-sphere bounds a PL n-ball. The following Alexander's theorem says that $\mathbb{R}^3$ is irreducible.

Theorem 3.1 (Alexander's theorem)**.** *Every PL-embedded 2-sphere in $\mathbb{R}^3$ bounds a PL 3-ball.*

Theorem 3.2. *Let $h\colon B^k \times \mathbb{R}^n \to \mathbb{R}^3$ be a topological embedding where $n+k=3$ such that h is PL in a neighbourhood of the boundary $\partial(B^k \times \mathbb{R}^n)$, then there exists a (topological) isotopy h_t from h to an embedding h_1 such that*

(1) h_1 is PL on $B^k \times B^n \subset B^k \times \mathbb{R}^n$
(2) $h_t = h$ on $\partial(B^k \times \mathbb{R}^n)$ and $B^k \times (\mathbb{R}^n \setminus 2B^n)$ for all t.

As we shall see later, $B^k \times \mathbb{R}^n$ will be viewed as an open k-handle lies in a chart of an ambient manifold. The proof of the theorem relies on a number of lemmas, most of which are specific to 3-manifolds. First recall that a *PL-immersion* is a local PL-embedding. The next result is proved by Whitehead in 1961.

Lemma 3.3. *Every PL n-manifold ($n \leq 3$) with no compact, unbounded components admits PL immersions in $\mathbb{R}^n$.*

Indeed, we will only apply Lemma 3.3 to $\mathbb{T}^n \setminus *$ for $n \leq 3$ so one can also just quote results from Section 1 which gives explicit smooth immersions of the n-torus for all n hence PL-immersions. The proof of Lemma 3.3 is fairly combinatorial and relies on properties of simplicial complexes, so is very different in flavour compared to the explicit immersions of the tori in Section 1.

A 3-manifold is *1-connected at infinity* if every compact subset is contained in another with 1-connected complement.

Lemma 3.4. *Let M be a PL 3-manifold which is 1-connected at infinity and has compact boundary. Let K be a compact subset of the interior of M, then M contains a compact PL-submanifold A with $\partial A = \partial M \sqcup S^2 \subset M$ such that K is contained in the interior of A.*

Proof sketch. Without loss of generality we assume that $M \setminus K$ is simply-connected. Let N be a regular neighbourhood of K which is contained in finitely many simplices and $W = M \setminus \mathrm{int}N$ connected. Label the components of $\partial N = \partial W$ by $Q_1, \dots, Q_r$. Each component of $N \setminus K$ contains just one Q_i, for suppose Q_1 and Q_2 are in the same component, we can join then by two arcs, one in N and one in W, and this gives a non-trivial loop in $M \setminus K$. We label the component containing Q_i by C_i. We would like to do some modifications such that all induced maps $\alpha_i \colon \pi_1(Q_i) \to \pi_1(C_i)$ and $\beta_i \colon \pi_1(Q_i) \to \pi_1(W)$ become injective. Once we have done this, we can apply Van-kampen and conclude that all Q_i's are simply connected hence spheres. Once we've done this, we can tubing them together and add the tubes to N to make it into one single sphere and the theorem is proved.

To do this, let g_i denotes the genus of Q_i and define non-negative integers $c_1 = \sum g_i$ and $c_2 = \sum \mathrm{Max}(g_i - 1, 0)$. Suppose $\alpha_1(\beta_1)$ is not injective, then Dehn's lemma(see below, lemma3.9) provides an embedded disk in C_1(respectively W) meeting Q_1 at the boundary circle which is an non-trival element of $\pi_1(Q_1)$. Thicken D^2 up to a 3-cell meeting Q_1 at $S^1 \times I$, then we replace N by $N - D^2 \times \mathrm{int}I$(respectively by $N \cup D^2 \times \mathrm{int}I$. Now $\partial N = Q_1' \cup \cdots \cup Q_r$ with $Q_1' = Q_1 - (S^1 \times I) \cup (D^2 \times \partial I)$.

There are two cases: if S^1 is a separating curve, then c_2 decreases by 1; if S^1 is not separating, then c_1 decreases by 1. In any case, we can continue this procedure until all α_i and β_i's are injective. $\square$

We remark that the result clearly also holds in the smooth case.

Definition 3.5. A properly embedded connected surface S in a 3 manifold is called *incompressible* if it is not S^2 and has trivial normal bundle, and for each 2-disk D in M with $D \cap S = \partial D$, there exists a 2-disk D' in S with $\partial D = \partial D'$. The disk D is sometimes called a compressing disk.

Notice that some authors also exclude D^2 such that surgery on an incompressible surface only splits off a copy of S^2. But we will allow D^2 for our purpose.

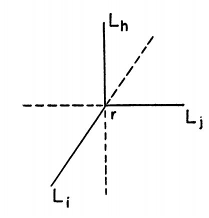
Definition 3.6. A PL 3-manifold M is called *sufficiently large* if it contains an incompressible surface.

A useful criteria of determining incompressible surface is the following: given a surface S other than S^2 with trivial normal bundle, if the induced map $\pi_1(S) \to \pi_1(M)$ on fundamental groups is injective, then S is incompressible. This is because every nullhomotopic circle in a surface bounds a disk. In fact, the converse is also true: suppose the induced map is not injective, let f be a null-homotopy of a non-trivial loop in S. We can deform f such that it is tranverse to S. The preimage $f^{-1}(S)$ consists of some circles which we can assume all non-trivial by redefine f if necessary. Then the restriction to the disk bounded by the inner most circle gives a null-homotopy of a non-trivial circle in S. Now Dehn's lemma(Lemma 3.9) gives a disk D in M with $D \cap S = \partial D$ and ∂D non-trivial in S. So S cannot be incompressible.

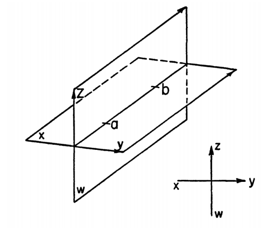
If we further require irreducibility then the manifold is called *Haken*. It is easy to see that $B^k \times \mathbb{T}^n$ is sufficiently large for $k = 0, 1, 2$ (for $k = 0, 1$, take the obvious embedded torus; for $k = 2$, take a properly embedded non-separating disk, for example, any standard disk bounded by a meridian in the solid torus).

Lemma 3.7. *Let M and N be orientable, compact, irreducible PL 3-manifolds with N sufficiently large and let $\phi \colon M \to N$ be a proper PL homotopy equivalence such that $\phi|_{\partial M}$ is a PL homeomorphism, then ϕ is homotopic relative boundary to a PL homeomorphism.*

The proof of this lemma is non-trival and involves the properties of incompressible surfaces in 3-manifolds and also properties of Haken manifolds, namely they have a hierarchy. Therefore, we will not go into the proof but just note that it can be generalised to the smooth case without much difficulty.



(A) Triple point



(B) Double curve

FIGURE 27. Two possible singularities

Lemma 3.8 (Alexander's isotopy: PL-version). *(1) If h_0 and h_1 are two PL-homeomorphisms of B^n that agree on the boundary S^{n-1}, then there exists a PL-isotopy h_t between them that fixes S^{n-1}.*
(2) Every PL-homeomorphism of S^{n-1} extends to a PL-homeomorphism of B^n.

Proof sketch. The second statement follows directly by coning. For the first one, notice that $B^n \times [-1, 1] \cong v * (S^{n-1} \times [-1, 1] \cup B^n \times \{-1, 1\})$ where $*$ denotes the join operation. Let $H: S^{n-1} \times [-1, 1] \cup B^n \times \{-1, 1\} \to S^{n-1} \times [-1, 1] \cup B^n \times \{-1, 1\}$ by $H|S^{n-1} \times [-1, 1] \cup B^n \times \{-1\}$ and $H|B^n \times \{1\} = h_1 h_0^{-1}$. Then apply coning. □

As a remark, in fact, this statement do hold in the smooth case for $n = 3$ but it's non-trivial. Indeed, we have $\mathrm{Diff}(S^n) \simeq O(n + 1) \times \mathrm{Diff}(D^n, \partial)$ and Smale and Cerf proved that actually $\mathrm{Diff}(D^3, \partial) \simeq \mathrm{Diff}(S^2) \simeq O(3)$. This is called the Smale conjecture. See Hatcher's survey [Hat12].

Lemma 3.9 (Generalised Dehn's lemma). *Let M be a connected orientable 3-manifold and $f : S \to M$ be a map from a sphere with n punctures with boundary circles $(C_1, \ldots, C_n)$ to M such that S is PL-embedded near its boundary. Then a non-vacuous subset of $T = \{C_1, \ldots, C_n\}$, say $(C_1, \ldots, C_r)$, $r \leq n$ constitute the boundary of an embedded surface S' agrees with S near T.*

Proof. (Sketch) We will only indicate a few ideas but not go into all details. First we claim without proof that under good conditions, $f(S)$ can be isotoped to be 'canonical', i.e., only have the following types of singularities: double curves and triple points. See Figure 27. For a proof, see lemma 3.2 of [Pap57].

For simplicity, we only show the case $n = 1$. First not that we can assume that M is compact and deformation retracts to $f(S)$. If not, take a subcomplex of M containing $f(S)$ and by subdivision if necessary and taking the union of the derived complexes containing all vertices in the boundary of $f(S)$, we can find a compact submanifold deformation retracts to $f(S)$.

Next, we show that the lemma is true if V has no 2-sheeted cover. By assumption, $H_1(V)$ is finite, otherwise we will have an induced surjective homomorphism from $\pi_1(v)$ to $\mathbb{Z}_2$ with an index 2 kernel. It follows from the universal coefficient theorem and Poincare duality that ∂V is a union of spheres so we are done.

Now suppose V has a 2-sheeted cover $p: V_1 \to V$ and let τ be the non-trivial deck transformation. Then $p^{-1}(C) = C_1 \cup \tau(C_1)$ where C_1 is a curve in V_1. It turns out that if C_1 satisfies the lemma for V_1, then C satisfies the lemma for V. To see this, let D_1 be an embedded disk in V_1 with boundary C_1 and let $D = P(D_1)$. We claim(without proof) that in this case D can be assumed to be canonical. Then since our cover is 2-sheeted, D can't have triple either so the only singularity we need to consider is double curve and one can avoid this but cutting along the double curves and analyse locally(again, details are in [Pap57]).

Now, let $d((f(S))$ denote the number of double curves and induct on d by taking double covers repeatedly, we have the result.

The proof of the general case is similar but more complicated and involves a calculation of the Euler characteristic and we omit here.([SW58]). □

Note that when $r = 1$ this reduces to the usual Dehn's lemma. Also, we remark that the proof works equally well in the smooth case.

Recall that a 3-manifold is called *prime* if it can not be written as a connected sum of two manifolds with neither of them is S^3. The next result is standard:

Lemma 3.10 (Prime decomposition theorem). *Every PL compact, orientable 3-manifold is a unique finite connected sum of prime 3-manifolds up to insertion or deletion of S^3's.*

Lemma 3.11. *For $(B^k \times \mathbb{T}^n)_\Sigma$ $(k = 0, 1, 2)$ where Σ is some PL structure coincides with the standard structure on $B^k \times B^n$, there exists a PL structure Σ' with $\Sigma' = \Sigma$ on $B^k \times B^n$ such that $(B^k \times \mathbb{T}^n)_{\Sigma'}$ is irreducible.*

Proof. By the prime decomposition theorem, $(B^k \times \mathbb{T}^n)_\Sigma$ is a connected sum of PL irreducible manifolds. But every PL 2-sphere in $B^k \times \mathbb{T}^n$ bounds a topological 3-ball (to see this, lift to the universal cover) so all but one prime factors are topological 3-spheres. Therefore, $(B^k \times \mathbb{T}^n)_\Sigma = A \cup Q$ where Q is a topological 3-ball with $A \cap Q = \partial Q \cong S^2$. Extend the identity map of A by coning gives a homeomorphism of $B^k \times \mathbb{T}^n$ and induces a PL structure Σ' with $(B^k \times \mathbb{T}^n)_{\Sigma'}$ irreducible and $\Sigma' = \Sigma$ on A. We will show that $B^k \times B^n$ can be assumed to be contained in A.

For $k = 0$, $(\mathbb{T}^3 \setminus B^3)_\Sigma$ is 1-connected at infinity so apply Lemma 3.4 to $\mathbb{T}^3 \setminus B^3$ gives a PL 2-sphere bounding a PL 3-ball containing B^3 in $\mathbb{T}^3$. Now choose the prime decomposition such that D is contained in A.

For $k = 1, 2$, the generalised Dehn's lemma provides k surfaces of type $(0, n)$ in $(B^k \times (2B^n \setminus B^n))_\Sigma$ with boundary $(\partial B^k \times \partial 1.5B^n)_\Sigma$. The union of the surface(s) and $(\partial B^k \times 1.5B^n)_\Sigma$ is a PL 2-sphere in $(B^k \times 2B^n)_\Sigma$ bounding a PL 3-ball D containing $(B^k \times B^n)_\Sigma$. Now choose a prime decomposition of $(B^k \times \mathbb{T}^n)_\Sigma \setminus D$ and reattach D to the corresponding summand, we get a desired decomposition. $\square$

We are now ready to prove the handle straightening theorem.

Proof of Theorem 3.2. Let $\mathbb{T}^n \setminus *$ be a punctured torus. Let $\Sigma = h^{-1}$(standard structure on $B^k \times \mathbb{R}^n$). For $k = 3$, $h\colon (B^3)_\Sigma \to \mathbb{R}^3$ is PL and by coning the identity map of ∂B^3 we get a PL homeomorphism $g\colon (B^3)_\sigma \to B^3$ that is identity near the boundary. Here we used the fact that h is PL near ∂B^3 so $(B^3)_\Sigma$ is standard near ∂B^3. By Lemma 3.8, we get an isotopy g_t from the identity to g. Then one checks that $hg_t{}^{-1}$ is the desired ambient isotopy.

For $k = 0, 1, 2$, we constructed a torus trick diagram as follows:

(1) Take an immersion ϕ_1 of $\mathbb{T}^n \setminus *$ in $\mathbb{R}^3$. Let $\alpha\colon B^k \times (\mathbb{T}^n \setminus *) \to B^k \times \mathbb{R}^n$ be the product of ϕ_1 and identity. By choosing our immersion carefully, we can assume that the bottom triangle of Figure 28 commutes. Define $\Sigma_1 = \alpha^{-1}(\Sigma)$. By construction, Σ_1 coincides with the standard structure on $B^k \times B^n$.

(2) Extend Σ_1 to $\partial B^k \times \mathbb{T}^n$ by letting it be the standard structure near an open collar $N(\partial B^k \times \mathbb{T}^n)$. Now $(B^k \times (\mathbb{T}^n \setminus *) \cup N(\partial B^k \times \mathbb{T}^n))_{\Sigma_1}$ is 1-connected at infinity, so by Lemma 3.4, it contains a compact PL submanifold K with boundary $(\partial B^k \times \mathbb{T}^n)_{\Sigma_1}$ and a 2-sphere S such that $B^k \times 2B^n$ is contained in its interior. By lifting to universal covers and apply the Schoenflies theorem, S bounds a topological 3-ball in $B^k \times \mathbb{T}^n$. Extend the identity map of K by coning over S gives a homeomorphism of $B^k \times \mathbb{T}^n$ which induces a PL structure Σ_2 on $B^k \times \mathbb{T}^n$. Note that since K is compact, coning must fill up all of $B^k \times \mathbb{T}^n$. By Lemma 3.11, we may assume that $(B^k \times \mathbb{T}^n)_{\Sigma_2}$ is irreducible. By applying simplicial approximation to the identity map $(B^k \times \mathbb{T}^n)_{\Sigma_2} \to B^k \times \mathbb{T}^n$ and apply Lemma 3.7, the identity map is homotopic relative to boundary to a PL homeomorphism g as in Figure 28.

(3) Pull Σ_2 back to a PL structure Σ_3 on $B^k \times \mathbb{R}^n$ via the universal covering map. By arranging the inclusion $B^k \times 2B^n$ appropriately we can make sure every thing still commutes. Lift g to a Pl homeomorphism $\widetilde{g}$ which is identity on the boundary. By lemma 10.5 in the lecture notes, $\widetilde{g}$ has bounded distance from identity.

(4) Let $\gamma\colon B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n$ be a PL embedding that maps onto $(B^k \times 2B^n) \setminus \{0\} \times \partial 2B^n$ and restricts to identity on $B^k \times B^n$.(This is very similar to what Hatcher did for surfaces). Let

$$\begin{tikzcd}
& (B^k \times \mathbb{R}^n)_{\Sigma_4} \arrow[r, "G"] & B^k \times \mathbb{R}^n \\
& (B^k \times \mathbb{R}^n)_{\Sigma_3} \arrow[u, hook, "\gamma"] \arrow[d, "e"] \arrow[r, "\tilde{g}"] & B^k \times \mathbb{R}^n \\
& (B^k \times T^n)_{\Sigma_2} \arrow[r, "g"] & B^k \times \mathbb{T}^n \\
& (B^k \times T^n - *)_{\Sigma_1} \arrow[u, hook] \arrow[d, "\alpha"] & \\
(B^k \times 2B^n) + \Sigma \arrow[r, hook] \arrow[uuuur, hook] \arrow[uuur, hook] \arrow[uur, hook] \arrow[ur, hook] & (B^k \times \mathbb{R}^n)_{\Sigma} \arrow[r, "h"] & \mathbb{R}^3
\end{tikzcd}$$

FIGURE 28. Torus trick diagram

$G = \gamma\widetilde{g}\gamma^{-1}$ defined on $(B^k \times 2B^n) \setminus \{0\} \times \partial 2B^n$ and extend it by identity to a homeomorphism of $B^k \times B^n$ that is identity on the boundary.(Similar to the proof of theorem 19.1 of lecture notes). Extend G further by identity gives a homeomorphism of $B^k \times \mathbb{R}^n$. Define $\Sigma_4 = G^{-1}$(Standard structure). By construction, $\Sigma_4 = \Sigma_3$ on $B^k \times B^n$.

Now define an isotopy

$$G_t = \begin{cases} \text{Alexander isotopy from the identity to } G & \text{on } B^k \times 2B^n \\ \text{Id} & \text{Otherwise} \end{cases}$$

One checks that $hG_0^{-1} = h$, hG_1^{-1} is PL on $B^k \times B^n$ and $hG_t^{-1} = h$ on $(B^k \times R^n \setminus B^k \times 2B^n) \cup \partial B^k \times \mathbb{R}^n$(recall that $\widetilde{g}$ is identity on the boundary. Thus hG_t^{-1} is the desired isotopy. □

Note that if we replace the simplicial approximation theorem by a version of the smooth approximation theorem and apply all the smooth versions of our lemmas, we can prove a handle smoothing theorem as we did in part 2 for surfaces.

3.2 Triangulation of 3-manifolds

Theorem 3.12. *(1) Every topological 3-manifold M admits a PL-structure hence a triangulation.*
(2) If Σ_1 and Σ_2 are two PL-structures on M, there exists an ambient isotopy of M from identity to a PL-homeomorphism between M_{Σ_1} and M_{Σ_2}.

We will need a general fact from point-set topology.Recall that a topological space is called **normal** if every two disjoint closed sets of have disjoint open neighborhoods. Note that topological manifolds are normal(for example, one can check this by noticing that they are metrizable).

Lemma 3.13 (Shrinking lemma)**.** *Let X be a normal space and $\mathcal{U} = \{U_i\}$ be a locally finite open cover, then there exists another open cover $\mathcal{W} = \{W_i\}$ such that the closure of W_i is contained in U_i for all i.*

By the classification of surfaces(instead, one can also quote the results from Section 2), every topological 3-manifold with boundary admits a PL structure on a collar of its boundary. Moreover, if

Σ_1 and Σ_2 are two PL-structures on M, then every homeomorphism $f\colon M_{\Sigma_1} \to M_{\Sigma_2}$ is isotopic to one that is PL on a collar of ∂M(for example, by applying the isotopy extension theorem)

Proof of Theorem 3.12. We prove existence first. The idea is to build up a PL structure inductively by patching up the local PL structures in each chart. Let $\mathcal{U} = \{U_i\}$ be a locally finite(hence countable, since we assume that M is second countable hence Lindelöf, which gives us countability) cover. By the paragraph before this proof, the subset U_0 of boundary charts can be assumed to be PL compatible. Relabel the elements of U_0 as $\dots U_{-2}, U_{-1}, U_0$ and the rest charts by $U_1, U_2, \dots$.

We proceed by induction. Suppose a PL structure has been constructed on $V_r = \bigcup_{i=-\infty}^{i=r} U_i$ and let $V = U_{r+1} \cap V_r$ with the PL structure inherited from U_{r+1}. U_{r+1} intersects finitely many charts $\{U_i\}_{i\in I}$ where I is some indexing set. Apply Lemma 3.13, we can replace U_i by an open subset of U_i whose closure is contained in U_i for all $i \in I$ and get a refined cover $\mathcal{W} = \{W_i\}$. By triangulating V we get a handle decomposition of V_r. Let K be the union of all closed 3-simplices with non-empty intersection with $\bigcup_{i\in I\cap\{-\infty,\dots,r+1\}} \overline{W_i}$. Apply Theorem 3.2 to handles corresponding to K in the order of 0,1,2 and 3-handles, we get a homeomorphism h of V that is PL on K and identity out side a compact neighbourhood $N(K)$ of K. Then $\bigcup_{i=-\infty}^{i=r+1} W_i$ has a well-defined PL-structure inherited from V_r on $\bigcup_{i=-\infty}^{i=r} W_i$, from U_{r+1} on $V_{r+1} \setminus N(K)$, and from h on $W_{r+1} \cap V$.

For uniqueness, first isotope the identity map to a homeomorphism that is PL on some collar c of ∂M. Triangulate $M \setminus \partial M$ and subdivide such that every simplex is contained in some Σ_2-chart of M. This gives a handle decomposition such that each handle lies in a Σ_2-chart. Apply Theorem 3.2 to all 0-handles with non-empty intersection with $M \setminus c$ and we get an ambient isotopy that is identity on a smaller collar. Now do the same thing successively for higher dimensional handles and this gives the desired isotopy. $\square$

As we explained along the way, we could have done the whole proof in the smooth case: Lemma 3.7, Lemma 3.8, Lemma 3.9, Lemma 3.10 hold in the smooth category. Furthermore, Lemma 2.5 can be easily generalised to dimension 3 with a similar proof, so our proof of Theorem 3.12 can be easily modified to a smooth version. However, Lemma 3.8 is a relatively deep result in the smooth case so this approach is not necessarily an easy one. Instead, one can construct directly a smooth structure for a PL 3-manifold by defining a version of tangent space for PL manifolds called wieldings. In summary, we now have an understanding of the sentence: for dimension lower or equal than 3, PL, smooth and topological categories are equivalent. Note that in dimension 4, this version of the handle straightening theorem must not true because of the known exotic phenomenons.

Remark 3.14. In dimensions ≤ 7, every PL-structure can be upgraded to a smooth structure, and for dimension ≤ 6 this associated smooth structure is unique up to isotopy, [Mil11, Thm. 2].

Further reading

– Andrew Ranicki's slides: 'High dimensional manifold topology, then and now' (2005)
– Lurie's lecture notes on Whitehead's theorem that smooth manifolds admit PL triangulations

References

[Bud] Ryan Budney. Immersing punctured torus. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/337288 (version: 2013-03-21).
[Fer74] Steven Ferry. An immersion of $T^n - D^n$ into R^n. *Enseign. Math. (2)*, 20:177–178, 1974.
[Gra74] André Gramain. Construction explicite de certaines immersions de codimension 0 ou 1. *Enseign. Math. (2)*, 20:333–337, 1974.
[Ham76] A. J. S. Hamilton. The triangulation of 3-manifolds. *Quart. J. Math. Oxford Ser. (2)*, 27(105):63–70, 1976.
[Hat12] Allen Hatcher. A 50-year view of diffeomorphism groups., 2012.
[Hat13] Allen Hatcher. The Kirby torus trick for surfaces, 2013.
[KS77] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[Mil97] John W. Milnor. *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.

[Mil11] John Milnor. Differential topology forty-six years later. *Notices Amer. Math. Soc.*, 58(6):804–809, 2011.

[Pap57] C. D. Papakyriakopoulos. On dehn's lemma and the asphericity of knots. *Annals of Mathematics*, 66(1):1–26, 1957.

[Rus73] T. Benny Rushing. *Topological embeddings*. Academic Press, New York-London, 1973. Pure and Applied Mathematics, Vol. 52.

[SW58] Arnold Shapiro and J. H. C. Whitehead. A proof and extension of Dehn's lemma. *Bulletin of the American Mathematical Society*, 64(4):174 – 178, 1958.

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