MODIFIED SURGERY || TOPOLOGY LEARNING SEMINAR

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Contents

1. Introduction

Modified surgery is a theory for classifying manifolds. It was modified, by Matthias Kreck, from the classical surgery theory of Kervaire-Milnor, Browder, Novikov, Sullivan, and Wall. The modified approach has advantages over the classical approach, and some disadvantages. But overall it is a very useful approach, and a practical theory that one can use to prove some great theorems.

Let CAT be one of TOP, PL, or DIFF. When we say we are classifying manifolds, we think of the situation that we are given two CAT n-manifolds, and we want to give a procedure, or list of algebraic invariants, that can decide whether the two manifolds are CAT-equivalent. Here TOP equivalent means homeomorphic, PL equivalent means PL-homeomorphic, and DIFF-equivalent means diffeomorphic.

The key breakthrough in the theory of the classification of manifolds of dimension at least 4 was the h-cobordism theorem. This is mostly due to Smale, with contributions by Stallings, Kirby-Siebenmann, and Freedman-Quinn.

Let M_1 and M_2 be closed, connected n-dimensional manifolds. A *cobordism* W is a compact $(n+1)$-dimensional manifold with $\partial W = M_1 \sqcup M_2$. A cobordism $(W; M_1, M_2)$ is an *h-cobordism* if the inclusion maps $i_1 \colon M_1 \to W$ and $i_2 \colon M_2 \to W$ are homotopy equivalences.

Theorem 1.1 (h-cobordism theorem)**.** *Let* $(W; M_1, M_2)$ *be a* CAT *h-cobordism between simply-connected* CAT *n-manifolds, with $n \geq 4$. If $n = 4$ then assume*

CAT = TOP. *Then there exists a* CAT*-equivalence* $F\colon W \xrightarrow{\cong} M_1 \times I$ *such that*

$$\begin{array}{ccc} M_1 & \xrightarrow{i_1} & W \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong}\,{\scriptstyle F} \\ M_1 \times \{0\} & \longrightarrow & M_1 \times I \end{array}$$

commutes.

In particular, $F|_{M_2}\colon M_2 \to M_1$ is a CAT-equivalence. So the h-cobordism theorem is extremely useful.

Here is a famous corollary.

Corollary 1.2 (Poincaré conjecture)**.** *Let M be a closed, smooth or* PL *n-manifold, for some $n \geq 6$, with $M \simeq S^n$. Then M is* PL*-homeomorphic to S^n.*

Proof. Remove two open balls. The remainder is an h-cobordism, therefore is diffeomorphic/PL-homeomorphic to the product $S^{n-1} \times I$, restricting to Id on one boundary compenent. Consider this as a PL-homeomorphism. Glue back in the balls and extend the homeomorphism over $D^n \sqcup D^n$ by coning. □

The fact that coning cannot be done smoothly for the second copy of D^n is why this proof does not work in the smooth category.

In fact there is also a version of the h-cobordism theorem without the simply connected hypothesis. A cobordism $(W; M_1, M_2)$ is an *s-cobordism* if the inclusion maps $i_1\colon M_1 \to W$ and $i_2\colon M_2 \to W$ are simple homotopy equivalences.

Theorem 1.3 (s-cobordism theorem)**.** *Let $(W; M_1, M_2)$ be a* CAT *s-cobordism between* CAT *n-manifolds, with $n \geq 4$. If $n = 4$ then assume* CAT = TOP *and that $\pi_1(W)$ is a good group. Then there exists a* CAT*-equivalence* $F\colon W \xrightarrow{\cong} M_1 \times I$ *such that*

$$\begin{array}{ccc} M_1 & \xrightarrow{i_1} & W \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong}\,{\scriptstyle F} \\ M_1 \times \{0\} & \longrightarrow & M_1 \times I \end{array}$$

commutes.

So the general principle still applies, there are just some extra hypotheses in the non-simply connected case.

The strategy of modified surgery is as follows.

(1) Start with two n-manifolds M_1 and M_2.
(2) Construct a cobordism between them.
(3) Attempt to perform surgery on the cobordism, away from the boundary, to convert it into an s-cobordism.

Along the way we might find obstructions, and we attempt to interpret these in terms of computable invariants of M_1 and M_2. Here is one of our goal applications for the term.

Theorem 1.4 (Freedman-Quinn). *Let M_1 and M_2 be topological, closed, connected, simply-connected 4-manifolds. Suppose that there is an isometry*

$$F\colon H_2(M_1) \to H_2(M_2)$$

between the intersection forms

$$\lambda_{M_i}\colon H_2(M_i) \times H_2(M_i) \to \mathbb{Z}$$

$i = 1, 2$. If M_1 and M_2 are both not spin suppose that the Kirby-Siebenmann invariants are equal: $\mathrm{ks}(M_1) = \mathrm{ks}(M_2) \in \mathbb{Z}/2$.

Then there is a homeomorphism

$$f\colon M_1 \xrightarrow{\cong} M_2$$

such that $f_* = F\colon H_2(M_1) \to H_2(M_2)$, which is unique up to isotopy.

Perhaps Csaba will also teach us something about complete intersections.

2. Normal k-smoothings

Let B have the homotopy type of a CW-complex and let $(M, \partial M)$ be an n-dimensional manifold properly embedded in $(\mathbb{R}^+ \times \mathbb{R}^N, \{0\} \times \mathbb{R}^N)$ for some sufficiently large $N >> 2n$. This embedding defines a Gauss map $\nu_M\colon M \to BO$. (To define BO, take the Grassmann manifold $G(n, k)$, the space of all n-planes intersecting the origin in $\mathbb{R}^{n+k}$, and define $BO(n)$ as the union $\bigcup_k^\infty G(n, k)$. Since we have the natural inclusion $BO(n) \hookrightarrow BO(n+1)$ given by setting the $(n+1)$-th coordinate to zero, we can define BO as the colimit of $BO(n)$ as $n \to \infty$.)

Assume we can lift ν_M to a map $\overline{\nu}_m\colon M \to B$. To be more precise, assume we have the following commutative diagram (up to homotopy) where the map $B \to BO$ is a fibration:

$$\begin{array}{ccc} & & B \\ & {\scriptstyle \overline{\nu}_M} \nearrow & \downarrow \\ M & \xrightarrow[\nu_M]{} & BO \end{array}$$

We then consider this lift up to homotopy to give us a well defined homotopy class of homotopy lifts i.e. the above diagram commutes up to homotopy.

Now suppose we have some different embedding $(M, \partial M) \hookrightarrow (\mathbb{R}^+ \times \mathbb{R}^N, \{0\} \times \mathbb{R}^N)$ which comes with a different Gauss map ν'_M. Since N is sufficiently large, we know that these two embeddings are isotopic, and this isotopy induces a homotopy from ν_M to ν'_M. This homotopy then induces a homotopy on the lift $\overline{\nu}_M$, taking it to a lift of ν'_M. This gives us a well-defined map

$$\left\{ \begin{array}{l} \text{homotopy classes of} \\ \text{homotopy lifts of } \nu_M. \end{array} \right\} \to \left\{ \begin{array}{l} \text{homotopy classes of} \\ \text{homotopy lifts of } \nu'_M. \end{array} \right\}$$

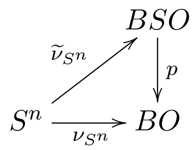
Since we have picked N to be sufficiently large, we have that all isotopies are themselves isotopic to one another. This means that the above map is a bijection, and hence we can consider lifts as equivalence classes under this map for all possible embeddings. In other words, this we can consider lifts independently of the choice of embedding for M. This allows us to make the following definition.

Definition 2.1. A *B-structure* on M is an equivalence class of homotopy classes of homotopy lifts of ν_M for some embedding of M. A pair $(M^n, \overline{\nu}_{M^n})$ of an n-manifold together with a B-structure $\overline{\nu}_{M^n} \colon M \to B$ is called an *n-dimensional B-manifold.*

One may wonder: given two representatives for B-structures $\overline{\nu}_M$ and $\overline{\nu}'_M$, how can we tell if they represent the same B-structure since we have suppressed the data about the embedding? The answer is that we can recover the map ν_M as $\overline{\nu}_M$ post-composed with the projection $B \to BO$. We know that this is realised as the stable normal bundle of some embedding M, and we can do the same procedure for ν'_M to get a different embedding of M. Now we have recovered all of the information about the embeddings and can check if these two B-structures are equivalent under the map defined above.

Here is a simple example of a B-structure and B-manifold.

Example 2.2. Let $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ be the unit sphere. We have the Gauss map $\nu_{S^n} \colon S^n \to BO$ defined by this standard embedding. Since S^n is orientable, we can lift this map to $\overline{\nu}_M$ such that we have the diagram:

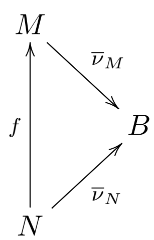


where p is the function that 'forgets' the orientation. (BSO is defined in much the same way as BO, except we start with the oriented Grassmann manifold $\widetilde{G}(n,k)$ consisting of n-planes in $\mathbb{R}^{n+k}$ with a specified orientation.) Hence $(S^n, \overline{\nu}_{S^n})$ is a BSO-manifold. A choice of lift $\overline{\nu}_{S^n}$ is equivalent to a choice of orientation of the stable normal bundle of S^n, which determines and is determined by an orientation of S^n.

If $(M, \overline{\nu}_M)$ is a B-manifold, then:

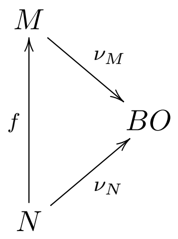
- $(\partial M, \overline{\nu}_M|_{\partial M})$ is a B-manifold.
- Under the identification $M = M \times \{0\}$, we can extend the B-structure of M onto the manifold $M \times I$. We do this by first properly embedding $(M \times I, \partial(M \times I)) \hookrightarrow (\mathbb{R}^+, \mathbb{R}^N)$ such that the embedding restricted to $M \times \{0\}$ matches the given embedding for M and then extending the given lift $\overline{\nu}_M$ across the rest of $M \times I$. The induced lift restricted onto $M \times \{1\}$ is then used as the definition for $-\overline{\nu} := \overline{\nu}_{M \times I}|_{M \times \{1\}}$.

Definition 2.3. Let $(N, \overline{\nu}_N), (M, \overline{\nu}_M)$ be B-manifolds and let $f \colon N \to M$ be a diffeomorphism. We say that f is a *B-structure diffeomorphism* if the following diagram



commutes.

Note that the fact that f induces a B-structure on N is because the diagram



commutes.

Definition 2.4. Closed B-manifolds $(M, \overline{\nu}_M), (N, \overline{\nu}_N)$ are *B-bordant* if there exists a B-manifold $(W, \overline{\nu}_W)$ such that $\partial W = M \sqcup N$ and $\overline{\nu}_W|_M = \overline{\nu}_M$ and $\overline{\nu}_W|_N = \overline{\nu}_N$.

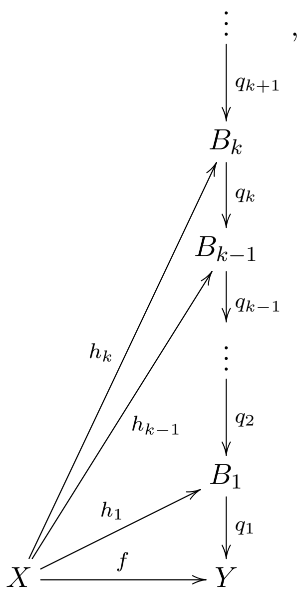
Proposition 2.5. *The set of B-manifolds up to B-bordism forms a group under disjoint union.*

The identity is, as usual, given by the empty set. The inverse of an element represented by $(M, \overline{\nu}_M)$ is represented by $(M, -\overline{\nu}_M)$, as these two B-manifolds form a B-bordism with the empty set. This explains the choice of notation. Associativity is clear.

We will now define a normal k-smoothing.

Definition 2.6. Let $(M, \overline{\nu}_M)$ be a B-manifold. We call $(M, \overline{\nu}_M)$ a *normal k-smoothing in B* if $\overline{\nu}_M$ is a $(k+1)$-connected map.

Definition 2.7. Let X and Y be CW-complexes and $f : X \to Y$ be a fibration with fibre F. A *Postnikov decomposition* of f is a commutative diagram



where each $q_k : B_k \to B_{k-1}$ is a fibration with fibre $K(\pi_k(F), k)$ and each $h_k : X \to B_k$ is a $(k+1)$-connected map.

There is an inductive arguement in both [add citations] which shows for any fibration we can construct a Postnikov tower given that the total space, base space and fibres are path connected.

We will often use the notion of a Postnikov tower of a map $f : X \to Y$ when f is not a fibration. What is meant in this case is to consider the pullback diagram

$$\begin{array}{ccc} W_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array},$$

where Y^I is maps of the interval into Y (do we want a basepoint)? and consider the map from $W_f \to Y$, which is a fibration and W_f is a homotopy equivalent to X.

All Postnikov decompositions of a fibration are unique up to homotopy equivalence, however Kreck uses the the term fibre homotopy equivalence so we will define this and provide a reference for a proof of these terms being equivalent for fibrations.

Definition 2.8. Let $p : D \to B$, $q : E \to B$ and $f : D \to E$ be a map such that $p = q \circ f$. We call f a *fibre homotopy equivalence* if their exists a g such that the following diagram

$$\begin{array}{ccc} D & \underset{g}{\overset{f}{\rightleftarrows}} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array},$$

commutes and their exist homotopies $F : D \times I \to D$ and $G : E \times I \to E$ such that $F_0 = g \circ f, F_1 = Id_D, G_0 = f \circ g$ and $G_1 = Id_E$ and $\forall t \in I$ the diagram

$$\begin{array}{ccc} D & \underset{G_t}{\overset{F_t}{\rightleftarrows}} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

commutes.

In the case where p and q are fibrations we do not need to think about this extra condition about the fibres of the D and E.

Proposition 2.9. *Let $p\colon D \to B$ and $q : E \to B$ be fibrations and $f\colon D \to E$ be a homotopy equivalence. Then f is a fibre homotopy equivalence.*

Hence, we can forget about fibre homotopy equivalence and only work with homotopy equivalence.

Theorem 2.10. *All Postnikov decompositions of a fibration are unique up to homotopy equivalence provided that the total space, base space and fibre are all path connected .*

Let B_k have homotopy type of a CW-complex and let $B_k \to BO$ be a $(k+1)$-co-connected fibration i.e. $\pi_k(F) = 0$ for $r \geq k+1$, where B_k and F are path connected. From the uniqueness of Postnikov decomposition, if we consider the map $M \xrightarrow{\nu_M} BO$ and the fibration $B_k \to BO$ coming from the Postnikov tower, such that it satisfies the above, then the fibre is an invariant of the manifold M.

Definition 2.11. We call this fibre the normal k-type of M.

Lemma 2.12. *Let $(M, \overline{\nu}_M)$ be a B-manifold. Then $(M, \overline{\nu}_M)$ is a k smoothing if and only if the triangle*

$$\begin{array}{ccc} & & B_k \\ & \overset{\overline{\nu}_M}{\nearrow} & \downarrow \\ M & \xrightarrow{\nu} & BO \end{array}$$

can be fitted into a Postinkov decomposition diagram.

Proof. Clearly, if the diagram can be fitted into a Postnikov tower you have a k-smoothing as the map is automatically $k+1$ connected. Now assume we have a k-smoothing in B_k, $(M, \overline{\nu}_M)$. We know that a unique, up to fibre homotopy equivalence, Postnikov decomposition exists for $M \xrightarrow{\nu_M} BO$, call each space in the decomposition B'_i. We must show we can construct a map from B_k to B'_k which is a fibre homotopy equivalence so that we can fit the triangle into a Postnikov tower.[Not sure how to move forward but probably need to use that we have a $k+1$ connected map]. □

3. Surgery

We take a brief detour to define the standard surgery operation.

Definition 3.1. Let M^n be a manifold. We define *n-dimensional k-surgery on M* to be the process of removing an embedding $i\colon S^k \times D^{n-k} \hookrightarrow M$ and forming a new manifold

$$M' := (M \smallsetminus i(S^k \times D^{n-k})) \cup_{S^k \times S^{n-k-1}} (D^{k+1} \times S^{n-k-1}).$$

The idea of surgery is that we would like to have a process which allows us to modify a manifold by removing an element in one of its homotopy groups. To be more specific, let x be a generator for $\pi_k(M)$. We would like to find a manifold M' which has $\pi_k(M') = \pi_k(M)/\langle x \rangle$. This may not be possible in general, but the idea is that we can find conditions for where surgery will allow us to achieve this. When this is possible, we say that the surgery *kills x*. Most of the details on this will be excluded, but you can find references in [REFERENCES].

The main criteria that we will use to determine whether or not surgery can kill an element will be referred to as 'surgery below the middle dimension'.

Lemma 3.2. *Let M^n be a manifold and x a generator of $\pi_k(M)$ with $k < \lfloor n/2 \rfloor$. Then we can kill x via surgery provided that $(\nu_M)_*[x] = 0 \in \pi_k(BO)$.*

Proof. Since $k < 2n$, we may assume that x is represented by an embedding $i : S^k \hookrightarrow M$ using the Whitney embedding theorem. Then consider the pullback of the tangent bundle of M along i which satisfies

$$i^*(\tau_M) = \tau_{S^k} \oplus \nu_i$$

where we write τ_N for the tangent bundle of a manifold N and ν_i for the normal bundle for the embedding i in M. Since the tangent bundle of the sphere is stably trivial, we know

$$i^*(\tau_M \oplus \varepsilon) = \varepsilon^{k+1} \oplus \nu_i$$

where ε is some trivial line bundle. Now we add to both sides the pullback of the stable normal bundle of M, we get

$$i * (\tau_M \oplus \nu_M \oplus \varepsilon) = \varepsilon^{k+1} \oplus \nu_i \oplus i * (\nu_M)$$

which shows that ν_i and $i^*(\nu_M)$ are stable inverses. This means that $(\nu_i)_*[x] = 0 \in \pi_k(BO) \cong \pi_k(BO(n-k))$ (the final isomorphism exists as $k < 2n$. This allows us to conclude that we can represent x by a framed embedding $S^k \times D^{n-k} \hookrightarrow M$, which means we can kill it using surgery on that embedding. □

Definition 3.3. The *trace* of a k-surgery on M^n is the cobordism

$$W = M \times I \cup_f (D^{k+1} \times D^{n-k})$$

where $f : S^k \times D^{n-k} \hookrightarrow M \times \{1\}$ is the surgery data.

This means that surgery operations preserve the cobordism class of a manifold. For our purposes, it is also necessary to know that surgery preserves the B-bordism class of a manifold.

Proposition 3.4. *Let $(M, \overline{\nu}_M)$ be a normal l-smoothing and let $(M', \overline{\nu}'_M)$ be the result of a k-surgery on M with $k < \lfloor n/2 \rfloor$ and $k \leq l$. Then $(M', \overline{\nu}'_M)$ is also a k-smoothing and the trace of this surgery $(W, \overline{\nu}_W)$ is a B-bordism.*

Proof. The only issue we have we extending the B-structure across is on $S^k \times D^{n-k} \subset M \times \{1\} \subset W$ where we have two competing B-structures, one induced by the B-structure on M and the other induced by the unique B-structure on $D^{k+1} \times D^{n-k}$.

[REFERENCE HERE or come up with an argument] We claim that these two structures can be made to agree and hence the trace is a B-bordism and since $\overline{\nu}_M$ is $(l+1)$-connected M' must also be a normal l-smoothing. □

4. Stable classification of $(n-1)$-smoothings of $2n$-manifolds

We now turn to an application of the theory that we developed in the previous section.

Definition 4.1. Let M and N be $2n$-manifolds. We say M and N are *stably diffeomorphic* if there exists k, l non-negative integers such that

$$M \#_k (S^n \times S^n) \cong N \#_l (S^n \times S^n).$$

Exercise 4.2. Show that if M and N are stably diffeomorphic as above then

$$k - l = (-1)^{n+1} \frac{\chi(M) - \chi(N)}{2}.$$

We would like to develop this notion for B-manifolds, so we need to construct a 'standard' set of B-structures on $S^n \times S^n$.

Definition 4.3. Let α be a B-structure on S^n. We say that α is *elementary* if $\alpha_*[S^n] = 0 \in \pi_n(B)$.

Note that the above definition does not determine a unique B-structure as it can be changed up to a choice of framing for S^n.

Given an elementary B-structure on S^n, we can extend this onto a B-structure for $S^n \times D^{m+1}$ via pre-composing α with the map projecting onto the first coordinate. Restricting this composition onto the boundary gives us a B-structure on $\partial(S^n \times D^{m+1}) = S^n \times S^m$. We will call such B-structures elementary also.

Definition 4.4. Let $(M, \overline{\nu}_M)$ and $(N, \overline{\nu}_N)$ be $2n$-dimensional B-manifolds. We say $(M, \overline{\nu}_M)$ and $(N, \overline{\nu}_N)$ are *stably diffeomorphic* if there exists k, l non-negative integers and $\alpha_1, \ldots, \alpha_k$; $\beta_1, \ldots, \beta_l$ elementary B-structures on $S^n \times S^n$ such that

$$(M, \overline{\nu}_M)\#(S^n \times S^n, \alpha_1)\# \ldots \#(S^n \times S^n, \alpha_k) \cong (N, \overline{\nu}_N)\#(S^n \times S^n, \beta_1)\# \ldots \#(S^n \times S^n, \beta_l).$$

Of course we can also make the exact same definition as above with B-structure diffeomorphism replaced with s-cobordism, and in that case we say that the B-manifolds are *stably s-cobordant*. In fact, this is the notion of equivalence that we will consider more often. Also, by the s-cobordism theorem, for $n \geq 2$ stably s-cobordant implies stably diffeomorphic.

We then consider the set of stable s-cobordism classes of (relative) $2n$-dimensional normal $(n-1)$-smoothings in B, denoted as NSt_{2n}^B. (If we are in the relative case, we first need to fix maps to a fixed $(2n-1)$-dimensional B-manifold $(V, \overline{\nu}_V)$ $f : \partial M \to V$ and we denote this set as $NSt_{2n}^{(B,V)}$.)

There is a map

$$\Phi \colon NSt_{2n}^B \to \Omega_{2n}^B$$

defined by taking a representative of a class on the left and B-bordism class on the right. Note that this map is well-defined because $(S^n \times S^n, \alpha)$ B-bounds $(S^n \times D^{n+1}, \alpha)$ and hence maps to zero.

Theorem 4.5. *The map Φ as defined above is a bijection for $n \geq 2$.*

To prove this we will need the use of a lemma which we state and prove now.

Lemma 4.6. *Suppose B has finite $\lfloor n/2 \rfloor$-skeleton. Then any B-manifold $(M^n, \overline{\nu}_M)$ is B-bordant to $(M', \overline{\nu}_{M'})$ such that $\overline{\nu}_{M'}$ is an $\lfloor n/2 \rfloor$-equivalence.*

Proof. Let X be the finite $\lfloor n/2 \rfloor$-skeleton of B and thicken it up to a compact $(n+1)$-dimensional manifold NX by replacing cells with handles and choosing framings such that the below diagram commutes.

$$\begin{array}{ccc} X & \xhookrightarrow{i} & B \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow \\ NX & \xrightarrow[\overline{\nu}_{NX}]{} & BO \end{array}$$

Here i is the inclusion into B which is an $\lfloor n/2 \rfloor$-equivalence and ι is the inclusion into NX which is a homotopy equivalence. This means that NX has a B-structure and it is an $(\lfloor n/2 \rfloor - 1)$-smoothing. Let $N := \partial NX$, which is an $(\lfloor n/2 \rfloor - 1)$-smoothing which B-bounds. In other words, $\overline{\nu}_N$ is surjective on homotopy groups π_k for all $k \leq \lfloor n/2 \rfloor$.

We can now take the B-manifold $(M', \overline{\nu}_{M'}) := (M, \overline{\nu}_M)\sharp(N, \overline{\nu}_N)$ which is B-bordant to $(M, \overline{\nu}_M)$ with $\overline{\nu}_{M'}$ surjective on homotopy groups π_k for all $k \leq \lfloor n/2 \rfloor$. To finish this proof, we need to kill the kernels of these induced maps via surgery.

Assume inductively that we have already killed the kernel of the map

$$(\overline{\nu}_{M'})_* \colon \pi_r(M') \to \pi_r(B)$$

for all $r < m < \lfloor n/2 \rfloor$. We want to kill the kernel for π_m. We have the long exact sequence of homotopy groups

$$\cdots \to \pi_{m+1}(M') \to \pi_{m+1}(B) \xrightarrow{0} \pi_{m+1}(B, M') \to \pi_m(M') \to \pi_m(B) \xrightarrow{0} \pi_m(B, M) \to \cdots$$

where the denoted zero maps are due to the surjectivity assumption. This means we can look at the following short exact sequence instead.

$$0 \longrightarrow \pi_{m+1}(B, M') \longrightarrow \pi_m(M') \xrightarrow{(\overline{\nu}_{M'})_*} \pi_m(B) \longrightarrow 0$$

By exactness the kernel of $(\overline{\nu}_{M'})_*$ is isomorphic to $\pi_{m+1}(B, M')$, which is itself isomorphic to $H_{m+1}(B, M'; \mathbb{Z}[\pi_1(B)])$. We know this homology group is finitely generated by [REFERENCE HERE] and so we know the kernel we want to kill is also finitely generated.

Let $x \in \ker((\overline{\nu}_{M'})_*)$ be a generator. Since $\overline{\nu}_{M'}$ is a B-structure, this means that $(\nu_{M'})_*(x) = 0 \in \pi_m(BO)$, so by 3.2 we can kill this element via surgery. As the kernel was finitely generated, a finite sequence of surgery steps now leads to a new B-manifold M'' with $\overline{\nu}_{M''}$ inducing isomorphisms on π_r for $r \leq m$.

Note that here we always require that $r \leq m < \lfloor n/2 \rfloor$ as then $r + m < n$. This means that by general position, we can always choose our surgeries to miss the spheres which map surjectively onto $\pi_r(B)$ for $r \leq \lfloor n/2 \rfloor$ and hence our surgeries do not destroy the surjectivity assumption.

Hence we have constructed a B-manifold B-bordant to M whose B-structure is an $\lfloor n/2 \rfloor$-equivalence. □

Proof of 4.5. By 3.2 every B-bordism class in Ω^B_{2n} contains a representative which is an $(n-1)$-smoothing. This implies that Φ must be surjective. All that is left is to prove injectivity.

Assume we have $(n-1)$-smoothings $(M, \overline{\nu}_M)$ and $(N, \overline{\nu}_N)$ such that $\Phi((M, \overline{\nu}_M)) = \Phi((N, \overline{\nu}_N))$, i.e. these manifolds are B-bordant. We want to show that these manifolds are stably s-cobordant. Let $(W^{2n+1}, \overline{\nu}_W)$ be the B-bordism between our manifolds.

Since $\overline{\nu}_W|_M = \overline{\nu}_M$ is surjective on $\pi_r(M) \to \pi_r(B)$ for $r \leq n$ we know $\overline{\nu}_W$ is also surjective on π_r for $r \leq n$. We can then use the same argument from 3.2 to use surgery to kill the kernel of this map to turn $\overline{\nu}_W$ into an n-equivalence. To complete the proof we need to show that the $M \hookrightarrow W$, $N \hookrightarrow W$ are (simple) homotopy equivalences. For the purposes of this proof we will assume all homotopy equivalences are simple.

Let $S^n \times D^{n+1} \hookrightarrow W$ be an embedding. Then we can join M and $\partial(S^n \times D^{n+1})$ by a thickened $I \times D^{2n}$ that meets both M and $\partial S^n \times D^{n+1}$ transversely (at $\partial I \times D^{2n}$). Removing $(S^n \times D^{n+1} \cup I \times D^{2n})$ from W results in a cobordism W' between $M\sharp(S^n \times S^n)$ and N. If $S^n \times \{0\} \to W \xrightarrow{\overline{\nu}_W} B$ represents $0 = [S^n] \in \pi_n(B)$ then this means we have an elementary B-structure α on $S^n \times S^n$ and we have a B-bordism $(W', \overline{\nu}_W|_{W'})$ between $(M, \overline{\nu}_M)\sharp(S^n \times S^n, \alpha)$ and $(N, \overline{\nu}_N)$.

The aim is now to find a finite number of disjoint embeddings $S^n \times D^{n+1} \hookrightarrow W$ with $S^n \to W \to B$ zero in π_n such that the result of removing these joined to M or N gives an s-cobordism. We claim that $\pi_n(W, M)$ and $\pi_n(W, N)$ are finitely generated. We already know that $\pi_k(W, M) = 0$ for all $k < n$ and so by Poincaré duality and the Hurewicz theorem showing that these inclusions are homotopy equivalences is equivalent to showing that $\pi_n = 0$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_{n+1}(B, W) & & & & \\
 & & \downarrow & \searrow & & & \\
\pi_n(M) & \longrightarrow & \pi_n(W) & \longrightarrow & \pi_n(W, M) & \longrightarrow & 0 \\
 & \searrow & \downarrow & & & & \\
 & & \pi_n(B) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

By a diagram chase, one can see that for every generator x of $\pi_n(M, W)$ there exists a $y \in \pi_n(W)$ mapping to $0 \in \pi_n(B)$ and to $x \in \pi_n(M, W)$ under the above maps in the diagram. We crucially use here that the map from $\pi_n(M) \to \pi_n(B)$ is surjective. Thus, by 3.2 we can represent generators of $\pi_n(W, M)$ by disjoint embeddings $(S^n \times D^{n+1})$ in W.

Now $\overline{\nu}_{W'}$ is still an n-equivalene since we only removed elements in $\ker(\overline{\nu}_{W'})_*$. By Hurewicz followed by excision we have that

$$\begin{aligned}
\pi_n(W', M\sharp_k(S^n \times S^n) &\cong H_n(W', M\sharp_k(S^n \times S^n); \mathbb{Z}[\pi_1]) \\
&\cong H_n(W, M \cup_k ((S^n \times D^{n+1}) \cup (I \times D^{2n})); \mathbb{Z}[\pi_1]) \cong 0.
\end{aligned}$$

We don't yet know that $\pi_n(W', N) = 0$, which is the only obstruction to finishing the argument.

By the same argument for M we can kill elements of $\pi_n(W', N)$ to give a B-bordism between $M\sharp_k(S^n \times S^n)$ and $N\sharp_l(S^n \times S^n)$, which has the inclusion maps inducing homotopy equivalences. $\square$

5. Stable classification of 4-manifolds

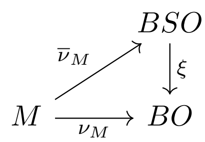
We now aim to look at some examples of applications of theorem 4.5.

Theorem 5.1. *Two closed smooth 4-manifolds M, N are stably diffeomorphic if and only if they have the same normal 1-type B and B-bordant normal 1-smoothings.*

We can make this more explicit by restricting to simply-connected 4-manifolds. Note that since all loops are trivial in this case our manifolds must be orientable. Let M be such a manifold. We now split into two cases, depending on whether $w_2(M) \neq 0$ or $w_2(M) = 0$.

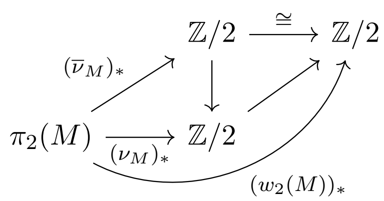
Case 1: $w_2(M) \neq 0$;

We claim that the normal 1-type is $B = BSO \xrightarrow{\xi} BO$ and summarise the information with the standard diagram



where ξ is the standard map induced on classifying spaces by the map that 'forgets' the orientation (compare to the example in 2.2). From the long exact sequence of homotopy groups it is easy to see that ξ is 2-co-connected.

We have that $\pi_1(M) \cong \pi_1(BSO) \cong 0$, so to be done all that we need to do is check that $\pi_2(\overline{\nu}_M)$ is surjective. We know that ξ has fibre at $K(\mathbb{Z}/2, 2)$ so on the π_2 level we have the diagram



Since $w_2(M) \neq 0$ and the diagram commutes we can conclude that $\overline{\nu}_M$ is 2-connected.

Case 2: $w_2(M) = 0$;

Now we claim that the normal 1-type is $B = B\mathrm{Spin} \xrightarrow{\xi} BO$. Because M is a spin manifold, we have the diagram

$$\begin{array}{ccc} & & B\mathrm{Spin} \\ & \overset{\overline{\nu}_M}{\nearrow} & \downarrow \xi \\ M & \xrightarrow[\nu_M]{} & BO \end{array}$$

where ξ is the composition of the maps induced on the classifying spaces by the covering map Spin $\to SO$ and the map that 'forgets' the orientation $SO \to BO$. Again, by looking at the long exact sequence of homotopy groups we can conclude that ξ is 2-co-connected, and since $\pi_2(B\mathrm{Spin}) = \pi_1(B\mathrm{Spin}) = 0$ we have automatically that $\overline{\nu}_M$ is 2-connected.

So if M and N have the same w_2 then we need to check whether they are B-bordant to conclude that they are stably diffeomorphic. Let us look at some bordism groups.

Case 1: $w_2(M) \neq 0$;

We want to consider the bordism group Ω_4^{SO} which is isomorphic to $\mathbb{Z}$. If we consider the topological category, we have the pair of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_4^{SO} & \longrightarrow & \Omega_4^{TOP} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \sigma \big\downarrow \cong & & (\sigma, \mathrm{ks}) \big\downarrow \cong & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

where the vertical maps are given by the signature σ and the Kirby-Siebenmann invariant, denoted as ks.

Case 2: $w_2(M) = 0$;

We instead consider the bordism group Ω_4^{Spin}, which again is isomorphic to $\mathbb{Z}$. This time we have the pair of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_4^{\mathrm{Spin}} & \longrightarrow & \Omega_4^{TOP\ \mathrm{Spin}} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \sigma \big\downarrow \cong & & \sigma \big\downarrow \cong & & & & \\ 0 & \longrightarrow & 16\mathbb{Z} & \longrightarrow & 8\mathbb{Z} & \longrightarrow & 8\mathbb{Z}/16\mathbb{Z} & \longrightarrow & 0 \end{array}$$

In conclusion, M, N are B-bordant (and hence stably diffeomorphic) if and only if they have the same signature (for some choice of orientations) and same w_2-type (both zero or both non-zero). In the topological category, we have to add to this that they must have the same Kirby-Siebenmann invariant. This completes the stable classification for simply-connected 4-manifolds.

What about if $\pi_1 \cong \mathbb{Z}$? We will go over this more briefly. Again, our normal 1-types depend on w_2 and are given by $B = S^1 \times BSO$ or $S^1 \times B\mathrm{Spin}$. This is easy to see by following the same arguments used for the simply-connected case. One has to argue that a 4-manifold with fundamental group $\mathbb{Z}$ that is not spin also has universal cover not spin, so there are only two cases, spin and non-spin.

We then need to consider the bordism groups. We write $\Omega_*(X) = \Omega_*^{SO}(X) = \Omega_*(X \times BSO) =$ to denote the $(X \times BSO)$-bordism group. We have

$$\begin{aligned}\Omega_4(S^1) &\cong \Omega_4 \oplus \widetilde{\Omega}_4(S^1)\\ &\cong \Omega_4 \oplus \widetilde{\Omega}_3(S^0)\\ &\cong \Omega_4.\end{aligned}$$

Here $\widetilde{\Omega}_*(X)$ denotes the reduced bordism group given as the kernel of the forgetful map $\Omega_*(X) \to \Omega_*(\mathrm{pt})$. The first isomorphism comes from the definition of reduced bordism. The second comes from using the suspension axiom for reduced bordism. The third isomorphism comes from using the definition of reduced bordism and the fact that $\Omega_3 \cong \Omega_3(\mathrm{pt}) = 0$.

Since $\Omega_3^{\mathrm{Spin}} = 0$ also, this same argument shows that $\Omega_4^{\mathrm{Spin}}(S^1) \cong \Omega_4^{\mathrm{Spin}}$, which means that the stable classification for 4-manifolds with $\pi_1 \cong \mathbb{Z}$ is exactly the same as for simply-connected 4-manifolds.

6. Cancellation of $S^n \times S^n$

Let $f : M\#rS^n \times S^n \to N\#r(S^n \times S^n)$ be a stable diffeomorphism of M and N. In this section we explain the conditions under which we can 'cancel' the $S^n \times S^n$ factors to obtain an s-cobordism between M ans N (diffeomorphism if $n > 2$). It turns out that the obstructions lie in the Whitehead group $\mathrm{Wh}(\pi_1(M))$ and in $L^s_{2n+1}(\pi_1(M), w_1(M))$.

We begin by stating the main theorem and some definitions and results. Then we give a brief explanation on why there are conditions on the dimension of the manifolds in the theorem.

Theorem 6.1. *Let $f : M\#rS^n \times S^n \to N\#r(S^n \times S^n)$ be a diffeomorphism and $n \neq 1, 3, 7$. If $\mathcal{V}(f) : H^r_{(-1)^n} \to H^r_{(-1)^n}$ is a simple ($\tau(f) = 0$) isometry and $[\mathcal{V}(f)] \in L^s_{2n+1}(\pi_1(M), w_1(M))$ vanishes then M and N are diffeomorphic under a diffeomorphism extending $f|_{\partial M} : \partial M \to \partial N$ (s-cobordant rel. boundary, if $n = 2$). If $n = 3$ or 7 and M is 1-connected we obtain the same conclusion if $\mathcal{V}(f)$ is an isometry of the intersection form.*

Definition 6.2. A *ring with involution* is a ring R with $- : R \to R$ satisfying $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\overline{a}, \overline{\overline{a}} = a$.

Let $\pi = \pi_1(M))$. Denote by $(\mathbb{Z}\pi, w)$ the group ring $\mathbb{Z}\pi$ with the w-twisted involution where w is the homomorphism from $\pi \to \{\pm 1\}$ sending a loop to ± 1 whether the orientation of the fibers are reversed or preserved (geometrically, it is the 1st Stiefel Whitney class). The w-twisted involution on $\mathbb{Z}\pi$ is defined by $* : \mathbb{Z}\pi \to \mathbb{Z}\pi$, $\sum\limits_{g\in\pi} n_g g \mapsto \sum\limits_{g\in\pi} w(g) n_g g^{-1}$.

Definition 6.3. Let R be a ring. A $(-1)^n$-*symmetric form* (V, λ) over R is a left R-module V together with a $(-1)^n$- hermitian form $\lambda : V \times V \to R$. That is, λ is linear in the first coordinate and $\lambda(v, w) = (-1)^n \overline{\lambda}(w, v)$.

Definition 6.4. A $(-1)^n$-quadratic form (V, λ, μ) over R is a symmetric form (V, λ) together with a *quadratic refinement* μ of λ, that is a map $\mu : V \to R/\langle x - (-1)^n \overline{x}\rangle$ satisfying:

- $\lambda(v, v) = \mu(v) + (-1)^n \overline{\mu}(v) \in R$.
- $\mu(v + w) = \mu(v) + \mu(w) + \lambda(v, w) \in R/\langle x - (-1)^n \overline{x}\rangle$.
- $\mu(x \cdot v) = x \cdot \mu(v) \cdot \overline{x}$.

Geometrically, λ and μ are given by the intersections and self-intersections of immersed spheres $S^n \looparrowright M^{2n}$. Denote by $I_n(M)$ the set of pointed regular homotopy classes of regular immersions $S^n \looparrowright M^{2n}$. It is an abelian group with addition by connected sums. And it inherits the structure of a $\mathbb{Z}\pi$-module coming from the action of π. The intersection pairing λ is defined by $\lambda : I_n(M) \times I_n(M) \to \mathbb{Z}\pi$, $\lambda(a, b) = \sum\limits_j \epsilon_j g_j$, where j runs over the intersection points between immersions a and b, ϵ_j is the sign of the intersection of a and b at j, and $g_j \in \pi$. The self intersection μ of $S^n \looparrowright M^{2n}$ is defined by $\mu : I_n(M) \to \mathbb{Z}\pi/\langle x - (-1)^n \overline{x}\rangle$, $\mu(a) = \sum\limits_j \epsilon_j g_j$, where j runs over the double points of the immersion a and $g_j \in \pi$. For a detailed treatment of intersection and self-intersection pairings, see [LCM]. Note that in this context μ is not a quadratic refinement of λ [LCM][Kreck2] for v, $w \in I_n(M)$, λ and μ satisfy the following:

- $\lambda(v, v) = \mu(v) + (-1)^n \overline{\mu}(v) + \chi(v) \in \mathbb{Z}\pi$, where $\chi(v)$ is the Euler number of the normal bundle ν_v.
- $\mu(v + w) = \mu(v) + \mu(w) + \lambda(v, w) \in \mathbb{Z}\pi/\langle x - (-1)^n \overline{x}\rangle$.
- $\mu(x \cdot v) = x \cdot \mu(v) \cdot \overline{x}$ for $x \in \pi$.

Denote by $K\pi_n(M)$ the kernel of the normal Gauss map $\pi_n(M) \to \pi_n(BO)$ and ν_α the normal bundle of the immersion $\alpha : S^n \looparrowright M$. A homotopy class $\alpha \in K\pi_n(M)$ is represented by an immersion with stably trivial normal bundle.

We will now give a brief explanation on why, for $n \neq 1, 3, 7$, every element in $K\pi_n(M)$ can be represented by a unique immersion with trivial normal bundle. And thus we get that μ is a quadratic refinement of λ and $(K\pi_n(M), \lambda, \mu)$ is a quadratic form. And we state what extra condition is needed for the cases $n = 3, 7$. We give a brief explanation which is based on [kreck2 p.727].

First, we state a few results, see [R, Propositions 5.55 , 11.24].

- The isomorphism class ν_f of the normal bundle of an immersion $f : S^n \looparrowright M^{2n}$ is a regular homotopy invariant.
- The geometric self intersection $\mu(f)$ of an immersion $f : S^n \looparrowright M^{2n}$ is a regular homotopy invariant.
- The connected sum of immersions $f : S^n \looparrowright M^{2n}$ and $f' : S^n \looparrowright M^{2n}$ is an immersion $f'' = f \# f' : S^n \# S^n \looparrowright M^{2n}$ with $\nu_{f''} = \nu_f + \nu_{f'}$.
- Let f, f', and f'' be as above. Then $\mu(f'') = \mu(f) + \mu(f') + \lambda(f, f')$, and $\chi(\nu_{f''}) = \chi(\nu_f) + \chi(\nu_{f'})$.

Theorem 6.5. *(Wall embedding Theorem) [R, 11.25]For $n \geq 3$, an immersion $f : S^n \looparrowright M^{2n}$ is regular homotopic to an embedding if and only if $\mu(f) = 0$*

Now let $M = M'\#S^n \times S^n$. There are two immersions representing the diagonal of $S^n \times S^n$, $\triangle : S^n \to S^n \times S^n$, $\triangle(x) = (x, x)$, and $q = q_1\#q_2$, where $q_1 : S^n \to S^n \times S^n$, $q_1(x) = (x, s)$, and $q_2 : S^n \to S^n \times S^n$, $q_2(x) = (s, x)$ for a base point s in S^n. Let $\triangle', q_1', q_2' : S^n \to M$ be the compositions of $\triangle$, q_1 and q_2 (which we assume miss the connected sum locus) with the inclusion map. Let $q' = q_1'\#q_2'$. These two immersions are not regularly homotopic since $\mu(\triangle') \neq \mu(q')$:
$\mu(\triangle') = 0$
$\mu(q') = \mu(q_1') + \mu(q_2') + \lambda(q_1', q_2') = [1]$

Now $\nu_{\triangle'}$ is isomorphic to TS^n [MS, Lemma 11.5] and $\nu_{q'} = \nu_{q_1'} + \nu_{q_2'} = 0$.

For $M^{2n}, n \geq 2$, there is an exact sequence

$$A \to I_n(M) \xrightarrow{f} \pi_n(M) \to 0$$

where $A = \mathbb{Z}$ if n is even and $A = \mathbb{Z}_2$ if n is odd [R, Proposition 7.39 and Example 7.40].

Consider the immersion $\triangle' - q'$. Since $\triangle'$ and q' represent the same element in $\pi_n(M)$, we get that $f(\triangle' - q') = 0$. The normal bundle of $\triangle' - q'$ is $TS^n = \nu_{\triangle'} - \nu_{q'}$.

We will start with the case n even.

Proposition 6.6. *A stably trivial n-dimensional, n even, bundle over S^n is trivial if and only if the Euler class vanishes and the group of these bundles are generated by the tangent bundle of S^n. (proof will be added)*

Let $\alpha \in K\pi_n(M)$. There is $g \in I_n(M)$ such that $f(g) = \alpha$. Note that $\chi(\nu_g)$ is even. Consider the immersion $g' = g\#\frac{\chi(\nu_g)}{2}(-(\triangle' - q'))$. Then, by the formula above, $\chi(\nu_{g'}) = \chi(\nu_g) + \frac{\chi(\nu_g)}{2}\chi(\nu_{-(\triangle'-q')}) = 0$ since $\chi(\nu_{\triangle'-q'}) = \chi(TS^n) = 2$. Hence, we can assume that an arbitrary immersion representing a homotopy class in $K\pi_n(M)$ can be represented by a unique immersion with trivial normal bundle. We can now define μ on $K\pi_n(M)$ and it follows from the formulas above that it is a quadratic refinement of λ and $\alpha \in K\pi_n(M)$ can be represented by an embedding with trivial normal bundle if and only if $\mu(\alpha) = 0$.

The case n odd, $n \neq 1, 3, 7$.

Proposition 6.7 (K-M, p.534)**.** *For n odd, $n \neq 1, 3, 7$, there are precisely two stably trivial bundles of dimension n over S^n, the trivial bundle and the tangent bundle of S^n.*

For $\alpha \in K\pi_n(M)$, either ν_α is trivial or $\nu_\alpha = TS^n$. Consider α with $\nu_\alpha = TS^n$. There is $g \in I_n(M)$ such that $f(g) = \alpha$. Let $g' = g\#(-(\triangle' - q'))$. Then $\nu_{g'} = 0$. We proceed as in the even case.

For $n = 3, 7$, n-dimensional stably trivial vector bundles over S^n are trivial. There is no unique immersion in $K\pi_n(M)$ with trivial normal bundle. Thus the self-intersection is not well defined in this case. However, if $\langle w_{q+1}(B), \pi_{q+1}(B)\rangle \neq 0$, then we can replace μ by the $\tilde{\mu}$ which takes values in $\mathbb{Z}\pi/\langle x - (-1)^n\overline{x}, 1\rangle$ and we get that $\alpha \in K\pi_n(M)$ can be represented by an embedding with trivial normal bundle if and only if $\tilde{\mu}(\alpha) = 0$. (needs elaboration).

Definition 6.8. The *standard $(-1)^n$-hyperbolic quadratic form* $H^r_{(-1)^n} := H_{(-1)^n}\perp\cdots\perp H_{(-1)^n}$, r summands, has underlying module $(\mathbb{Z}\pi\oplus\mathbb{Z}\pi)^r$ and basis $e_1,\cdots,e_r,f_1,\cdots,f_r$ such that $\lambda(e_i,e_j)=\lambda(f_i,f_j)=0$, $\lambda(e_i,f_j)=\delta_{ij}$, and $\mu(e_i)=\mu(f_i)=0$.

In our context, the restriction of $(\lambda,\mu)$to $r(S^n\times S^n)$ in $\pi_n(M)\#r(S^n\times S^n)$ is the standard hyperbolic quadratic form $H^r_{(-1)^n}$.

Definition 6.9. Two quadratic forms (V_1,λ_1,μ_1) and (V_2,λ_2,μ_2) are *isomorphic* if there is an isomorphism $f: V_1\to V_2$ such that $\lambda_2(f(x),f(y))=\lambda_1(x,y)$ and $\mu_2(f(x))=\mu_1(x)$.

Definition 6.10. An R-module V is *based* if it is finitely generated and equipped with an equivalenvce class of bases. Two bases are equivalent if the matrix of change of bases vanishes in $\mathrm{Wh}(\pi)$. An isomorphism between based R-modules is called *simple* if the matrix of the isomorphism with respect to the given bases vanishes in $\mathrm{Wh}(\pi)$.

Definition 6.11. The boundary connected sum $M^n\natural N^n$ of two manifolds with boundary is obrained by attaching a 1-handle $D^1\times D^{n-1}$ to the boundaries of M and N.

The condition in the theorem is that the composition

$$H^r_{(-1)^n}\to\pi_n(M)\oplus H^r_{(-1)^n}\xrightarrow{f_*}\pi_n(N)\oplus H^r_{(-1)^n}\to H^r_{(-1)^n}$$

which we denote by $\mathcal{V}(f)$ is an isometry. If $\mathcal{V}(f)$ is invertible, the Whitehead torsion, $\tau(f)$, in $\mathrm{Wh}(\pi)$ is defined and called the *Whitehead torsion of f on $r(S^n\times S^n)$*. If it vanishes, then $\mathcal{V}(f)$ is a simple isometry of $H^r_{(-1)^n}$ representing an element of $L^s_{2n+1}(\pi_1(M),w_1(M))$, which we denote by $[\mathcal{V}(f)]$.

Proof of the main theorem :

We prove the theorem for $n\neq 1,3,7$. If $[\mathcal{V}(f)]=0$ in $L^s_{2n+1}(\pi_1(M),w_1(M))$ then after further stabilisation we can assume that $\mathcal{V}(f)$ =Id. [] Consider the relative boundary bordism

$$M\times I\natural_r(S^n\times D^{n+1})\cup_f N\times I\natural_r(D^{n+1}\times S^n)$$

between M and N and write it as $X\cup_f Y$. We show that it is a relative s-cobordism. That is, we need:

- $\pi_1(M)\to\pi_1(X\cup_f Y)$ and $\pi_1(N)\to\pi_1(X\cup_f Y)$ are isomorphisms which are obvious.
- $H_*(X\cup_f Y,N;\mathbb{Z}\pi)=0$
- $(X\cup_f Y,N)$ has trivial Whitehead torsion.

We consider the homology exact sequence of triples $(X\cup_f Y,Y,N)$:

$$\cdots\to H_{n+1}(X\cup_f Y,N;\mathbb{Z}\pi)\to H_{n+1}(X\cup_f Y,Y;\mathbb{Z}\pi)\to H_n(Y,N;\mathbb{Z}\pi)\to H_n(X\cup_f Y,N;\mathbb{Z}\pi)\to\cdots$$

By excision $H_*(X\cup_f Y,Y;\mathbb{Z}\pi)=H_*(X,M\#r(S^n\times S^n);\mathbb{Z}\pi)=0$ if $*\neq n+1$ and $\mathbb{Z}\pi^r$ if $*=n+1$ and the disks $\{*\}\times D^{n+1}$ represent a preferred basis in dimension $n+1$. By construction $H_*(Y,N;\mathbb{Z}\pi)=0$ if $*\neq n$ and $\mathbb{Z}\pi^r$ if $*=n$ and the spheres $\{*\}\times S^n$ form a preferred basis in dimension n. By construction, the boundary map is $\mathcal{V}(f)$ restricted to the half rank subspace $(\{0\}\times\mathbb{Z}\pi)^r$. If $\mathcal{V}(f)$ =Id, it is a simple isomorphism and by the same argument as at the end of proposition [] it follows that $H_*(X \cup_f Y, N; \mathbb{Z}\pi) = 0$ and $(X \cup_f Y, N)$ has trivial Whitehead torsion.

7. Classification of closed simply connected 4-manifolds

Now we want to do better, and classify simply connected, closed 4-manifolds up to homeomorphism. This can be done using the classical surgery sequence, as explained in 'The Disc Embedding Theorem', and there is an argument in Freedman-Quinn's book. A third way is using Kreck's modified surgery, as we will explain here. For the stable classification, we used the normal 1-type. One can then either try to cancel $S^2 \times S^2$ summands (this does work, as shown by Hambleton and Kreck), or instead work with the normal 2-type. We will use the latter approach.

Theorem 7.1 (Freedman '82)**.** *(a) Let* $\lambda\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ *be a symmetric, bilinear, nonsingular form. Let* $k \in \mathbb{Z}/2$*. If* λ *is even then assume* $\operatorname{sign}(\lambda)/8 \equiv k \mod 2$*. Then there exists a closed, simply connected 4-manifold* X *with*

$$\lambda \cong \lambda_X\colon H_2(X) \times H_2(X) \to \mathbb{Z}$$

and $k = \operatorname{ks}(X) \in \mathbb{Z}/2$*.*

(b) Let X *and* Y *be two closed simply connected 4-manifolds with* $\operatorname{ks}(X) = \operatorname{ks}(Y) \in \mathbb{Z}/2$*. Let* $F\colon H_2(X) \to H_2(Y)$ *be an isomorphism inducing an isometry* $\lambda_X \cong \lambda_Y$*, i.e.* $\lambda_X(x,y) = \lambda_Y(F(x),F(y))$ *for all* $x, y \in H_2(X)$*. Then there is a homeomorphism* $\Phi\colon X \xrightarrow{\cong} Y$ *with* $\Phi_* = F$*.*

Our aim for these notes will be to prove (b) using modified surgery. We will focus on the case that X and Y are spin, since that will already take long enough to explain for two lectures.

For every simply connected 4-manifold with odd intersection form X, there is another 4-manifold $*X$ which is homotopy equivalent to X but has opposite Kirby-Siebenmann invariant.

Corollary 7.2. *If* X *is spin, and* Y *is a 4-manifold with* $Y \simeq X$*, then* $Y \cong X$*. If* X *is not spin, and* $Y \simeq X$*, then either* $Y \cong X$ *or* $Y \cong *X$*.*

Here is a summary of the proof.

(1) Find the normal 2-type B, and show that there are normal 2-smoothings from X and Y to B, over BSTOP (the analogue of BSO for topological normal bundles.
(2) Compute the bordism group of 4-manifolds over the normal 2-type. This uses that $\operatorname{ks}(X) = \operatorname{ks}(Y)$ and that $\lambda_X \cong \lambda_Y$.
(3) If X and Y are bordant, we have a bordism $(W; X, Y)$ with a map $\overline{\nu}_W : W \to B$. The modified surgery obstruction $\theta(W, \overline{\nu}_W)$ lives in $L_5(\mathbb{Z})$ (which is a subset of the ℓ_5 monoid). Since $L_5(\mathbb{Z}) = 0$, modified surgery theory implies that W is bordant rel. boundary, over B, to an h-cobordism W'. By the h-cobordism theorem, there is a homeomorphism $\Psi\colon W' \to X \times [0,1]$ restricting to the identity $X \to X \times \{0\}$. The restriction to Y yields a homeomorphism $Y \to X \times \{1\}$, and the inverse of this is the desired homeomorphism Φ.

(4) By keeping track of H_2 carefully, we can verify that $\Phi_* = F$.

So the key step is to show that X and Y have the same normal 2-type B and that they are bordant over B. The rest will then follow using the general theory we have been learning about.

Remark 7.3. If X and Y are smooth, then the same proof, with minor modifications to work over BSO instead of BSTOP, will show that X and Y are smoothly h-cobordant. But smooth h-cobordisms between 4-manifolds need not be trivial. The topological category is required at this point, so that we can appeal to the 5-dimensional h-cobordism theorem of Freedman and Quinn, and deduce that the 4-manifold are homeomorphic.

7.1. **The normal 2-type.** As mentioned above, there are two cases. We will focus on the spin case, i.e. when λ_X and λ_Y are even. Let

$$n := \beta_2(X) = \beta_2(Y).$$

We know the homology of X, by Poincaré duality and universal coefficients:

$$H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0, 4 \\ \mathbb{Z}^n & i = 2 \\ 0 & i = 1, 3. \end{cases}$$

Also $\pi_1(X) = 0$ and $\pi_2(X) \cong \mathbb{Z}^n$ by the Hurewicz theorem. It follows that the Postnikov 2-type of X is $K(\mathbb{Z}^n, 2) \simeq \prod^n \mathbb{CP}^\infty$. Fix an identification

$$\eta \colon \pi_2(X) \xrightarrow{\cong} \pi_2\Big(\prod^n \mathbb{CP}^\infty\Big) \cong \mathbb{Z}^n.$$

This determines a map $\eta \colon X \to \prod^n \mathbb{CP}^\infty$, up to homotopy. Let $\mathfrak{s} \colon X \to \mathrm{BTOPSpin}$ represent a spin structure on X. From now on write

$$K_n := \prod^n \mathbb{CP}^\infty.$$

Lemma 7.4. *Let X be a closed, simply connected, spin, topological 4-manifold with $\beta_2(X) = n$. With K_n, $\mathfrak{s}$, and η as above, let $B := \mathrm{BTOPSpin} \times K_n$, let $\overline{\nu}_X = \mathfrak{s} \times \eta$. Let $\xi \colon B \to \mathrm{BSTOP}$ be given by projection to BTOPSpin followed by the canonical map to BSTOP. These fit into the following commuting diagram.*

$$\begin{array}{ccc} & & B \\ & \overset{\overline{\nu}_X := \mathfrak{s} \times \eta}{\nearrow} & \big\downarrow \xi \\ X & \xrightarrow[\nu_X]{} & \mathrm{BSTOP}. \end{array}$$

Moreover, $\xi \colon B \to \mathrm{BSTOP}$ is the normal 2-type of X, and $\overline{\nu}_X$ is a normal 2-smoothing.

Note that Y also satisfies the hypotheses of the lemma.

Proof. We check that ξ is 3-coconnected and that $\overline{\nu}_X$ is 3-connected. Since the diagram commutes by definition of a spin structure, this will complete the proof.

To prove that ξ is 3-coconnected, let $i \geq 3$. Recall that $B = \text{BTOPSpin} \times K_n$. Then

$$\begin{aligned}\pi_i(\text{BTOPSpin} \times K_n) &\cong \pi_i(\text{BTOPSpin}) \cong \pi_{i-1}(\text{TOPSpin}) \\ &\cong \pi_{i-1}(\text{STOP}) \cong \pi_i(\text{BSTOP}).\end{aligned}$$

It follows that $\xi\colon B \to \text{BSTOP}$ induces an isomorphism on π_i for all $i \geq 3$, and therefore is certainly 3-coconnected.

Now we consider the homotopy groups of B in the range $1 \leq i \leq 3$. First, $\pi_1(B) = 0$, so $\overline{\nu}_X$ induces an isomorphism on π_1. Then:

$$\pi_2(B) \cong \pi_2(\text{BTOPSpin}) \times \pi_2(K_n) \cong \pi_1(\text{TOPSpin}) \times \mathbb{Z}^n \cong \mathbb{Z}^n.$$

The map $\eta\colon X \to B$ induced an isomorphism $\eta\colon \pi_2(X) \cong \pi_2(B) \cong \mathbb{Z}^n$, and therefore so does $\overline{\nu}_X$. Finally $\pi_3(B) \cong \pi_2(\text{STOP}) \cong \pi_2(SO) = 0$. So $\overline{\nu}_X$ certainly induces a surjection on π_3. This completes the proof that $\overline{\nu}_X$ is 3-connected, which completes the proof of the lemma. □

7.2. **The bordism group.** Now that we have found the normal 2-types of X and Y, and have seen that they coincide, we want to compute the group

$$\Omega_4(B, \xi) \cong \Omega_4^{\text{TOPSpin}}(K_n).$$

The fact that we can identify the bordism group with a more standard spin bordism group over K_n means it is not too hard to compute. It can be understood using an Atiyah-Hirzebruch spectral sequence:

$$E^2_{p,q} = H_p(K_n; \Omega_q^{\text{TOPSpin}}) \Rightarrow \Omega_4^{\text{TOPSpin}}(K_n)$$

with d^2 differential

$$d^2_{p,q}\colon H_p(K_n; \Omega_q^{\text{TOPSpin}}) \to H_{p-2}(K_n; \Omega_{q+1}^{\text{TOPSpin}}).$$

Helpfully, this differential can be computed using Steenrod squares, as we shall explain below.

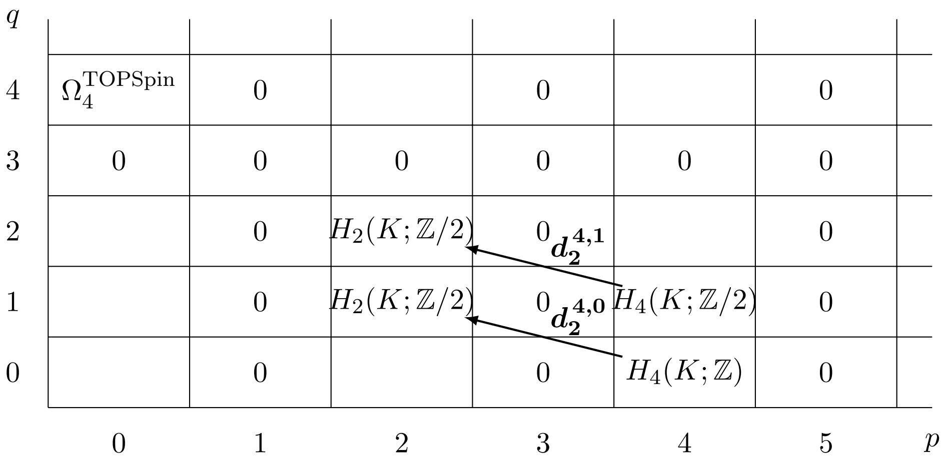
The convergence of the spectral sequence means that the E^∞ page terms with $p+q=4$ are the iterated graded groups in a filtration of $\Omega_4^{\text{TOPSpin}}(K_n)$ by subgroups. We need to draw the E_2 page. The first few spin bordism groups are as follows:

$$\Omega_q^{\text{TOPSpin}} \cong \begin{cases} \mathbb{Z} & q = 0, 4 \\ \mathbb{Z}/2 & q = 1, 2 \\ 0 & q = 3, 5. \end{cases}$$

The homology groups of K_n with coefficients in $R \in \{\mathbb{Z}, \mathbb{Z}/2\}$ in the range $0 \leq i \leq 5$ are:

$$H_i(K_n; R) \cong \begin{cases} R & i = 0 \\ R^n & i = 2 \\ R^{n(n+1)/2} & i = 4 \\ 0 & i = 1, 3, 5. \end{cases}$$

We can now draw the relevant groups in the E^2 page. In the diagram we write $K := K_n$.



We do not show the higher differentials, but note that they map $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$. All the groups in the spectral sequence outside the $p \geq 0, q \geq 0$ quadrant vanish.

First note that $\Omega_4^{\mathrm{TOPSpin}} \cong \mathbb{Z}$ is torsion-free, while all the groups that map to it are torsion or trivial. So $E^\infty_{0,4} \cong \mathbb{Z}$, and a 4-manifold maps in here to the signature of its intersection form.

We will use the next two propositions without proof. The rough idea is that naturality of spectral sequences sometimes can be used to deduce that differentials have to be related to nontrivial stable cohomology operations. In these low degrees the only such operation is the Steenrod square Sq^2.

Proposition 7.5. *Consider the Steenrod square* $\mathrm{Sq}^2\colon H^2(K_n;\mathbb{Z}/2) \to H^4(K_n;\mathbb{Z}/2)$. *The differential* $d^2_{4,1}\colon H_4(K_n;\mathbb{Z}/2) \to H_2(K_n;\mathbb{Z}/2)$ *fits into the following commutative diagram, where the vertical maps are given by the evaluation isomorphisms.*

$$\begin{array}{ccc} H_4(K_n;\mathbb{Z}/2) & \xrightarrow{\quad d^2_{4,1}\quad} & H_2(K;\mathbb{Z}/2) \\ \big\downarrow\cong & & \big\downarrow\cong \\ \mathrm{Hom}(H^4(K_n;\mathbb{Z}/2),\mathbb{Z}/2) & \xrightarrow{(\mathrm{Sq}^2)^*} & \mathrm{Hom}(H^2(K_n;\mathbb{Z}/2),\mathbb{Z}/2). \end{array}$$

The bottom row can be computed using cup products on K_n, which is a standard exercise in algebraic topology, using the Künneth theorem. Indeed,

$$H^*(K_n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,\dots x_n]$$

where each x_i is dual to a $\mathbb{CP}^1$ in one of the $\mathbb{CP}^\infty$s. We have

$$\begin{aligned} \mathrm{Sq}^2\colon H^2(K;\mathbb{Z}/2) &\to H^4(K;\mathbb{Z}/2) \\ x_i &\mapsto x_i^2. \end{aligned}$$

Therefore under the dual map $(\mathrm{Sq}^2)^*$, we see that $(x_i^2)^* \mapsto x_i^*$. So $d^2_{4,1}$ is onto. It follows that $E^3_{2,2} = E^\infty_{2,2} = 0$.

Proposition 7.6. *Consider the composition*

$$\mathrm{red}_2\colon H_4(K_n;\mathbb{Z}) \to H_4(K_n;\mathbb{Z}/2) \to \mathrm{Hom}(H^4(K_n;\mathbb{Z}/2),\mathbb{Z}/2)$$

of reduction mod 2 followed by the evaluation isomorphism. The differential

$$d^2_{4,0}\colon H_4(K_n;\mathbb{Z}) \to H_2(K_n;\mathbb{Z}/2)$$

fits into the following commutative diagram:

$$\begin{array}{ccc} H_4(K_n;\mathbb{Z}) & \xrightarrow{\quad d^2_{4,0}\quad} & H_2(K;\mathbb{Z}/2) \\ \big\downarrow \mathrm{red}_2 & & \big\downarrow \cong \\ \mathrm{Hom}(H^4(K_n;\mathbb{Z}/2),\mathbb{Z}/2) & \xrightarrow{(\mathrm{Sq}^2)^*} & \mathrm{Hom}(H^2(K_n;\mathbb{Z}/2),\mathbb{Z}/2). \end{array}$$

First, $\mathrm{Hom}(H^4(K_n;\mathbb{Z}/2),\mathbb{Z}/2)$ is generated by $(x_i^2)^*$ and $(x_ix_j)^*$, for $i,j = 1,\dots,n$. Since $\mathrm{Hom}(H^2(K_n;\mathbb{Z}/2),\mathbb{Z}/2)$ is generated by the x_i^*, for $i = 1,\dots,n$, we see that $\ker(\mathrm{Sq}^2)^* = \mathbb{Z}/2\langle (x_ix_j)^*\rangle_{i\neq j}$. We also have

$$H_4(K_n;\mathbb{Z}) \cong \mathbb{Z}\langle (x_i^2)^*\rangle \oplus \mathbb{Z}\langle (x_ix_j)^*\rangle.$$

We see that

$$\ker d^2_{4,0} = 2\mathbb{Z}\langle (x_i^2)^*\rangle \oplus \mathbb{Z}\langle (x_ix_j)^*\rangle.$$

This corresponds to the intersection form of a 4-manifold, the coefficient of (i,j) giving the corresponding entry of the matrix representing the form with respect to the basis $\{\mathrm{PD}\circ\eta^*(x_i)\}_{i=1}^n$. Here note that $n(n+1)/2$ integers suffices to determine all n^2 entries of the symmetric matrix. The fact that the diagonal entries are even corresponds to the fact that the manifold is spin.

Proposition 7.7.

$$\Omega_4^{\mathrm{TOPSpin}}(K_n) \cong \Omega_4^{\mathrm{TOPSpin}} \oplus \ker d^2_{4,0} \cong \mathbb{Z}\oplus \ker d^2_{4,0} \cong \mathbb{Z}\oplus\mathbb{Z}^{n(n+1)/2}.$$

Proof. The surviving terms on the E^∞ page are $\Omega_4^{\mathrm{TOPSpin}}$ and $\ker d^2_{4,0}$. This means that there is a filtration $0 \le F_{4,0} \le \Omega_4^{\mathrm{TOPSpin}}(K_n)$ with $\Omega_4^{\mathrm{TOPSpin}}(K_n)/F_{4,0} \cong \Omega_4^{\mathrm{TOPSpin}}$. In other words there is a short exact sequence

$$0 \to \ker d^2_{4,0} \to \Omega_4^{\mathrm{TOPSpin}}(K_n) \to \Omega_4^{\mathrm{TOPSpin}} \to 0.$$

Since $\Omega_4^{\mathrm{TOPSpin}} \cong \mathbb{Z}$, the sequence splits, and we have the claimed isomorphism. □

7.3. **Finishing the proof of (b) in the spin case.** Having computed the bordism groups, the remaining task is to show that $(X,\overline{\nu}_X)$ and $(Y,\overline{\nu}_Y)$ are equal in the bordism group. As explained in the summary, this will allow us to apply the general theory to complete the proof, because the resulting bordism over the 2-type has surgery obstruction in $L_5(\mathbb{Z}) = 0$, so it can be surgered to an h-cobordism. The h-cobordism theorem then yields the desired homomorphism.

Before we begin, we note that we have some freedom to choose the maps $\eta_X\colon X \to K_n$ and $\eta_Y\colon Y\to K_n$, provided they induce isomorphisms on H_2. Also recall that we are given an isometry $F\colon H_2(X)\to H_2(Y)$. Fix η_X, and then define η_Y so that

$$\eta_Y^* = \mathrm{PD}^{-1}\circ F\circ \mathrm{PD}\circ\eta_X^*\colon H^2(K_n)\to H^2(Y).$$

Since $H_2(Y)$ and $H_2(K_n)$ are torsion free and $H_1(Y) = 0 = H_1(K_n)$, the map $(\eta_Y)_*$, and whence the homotopy class ofη_Y, determines and is determined by the map $(\eta_Y)^*$. Note that we therefore have that

$$F \circ \mathrm{PD} \circ \eta_X^* = \mathrm{PD} \circ \eta_Y^* \colon H^2(K_n) \to H_2(Y).$$

We will use this below.

Now we proceed to show that, with the above choices of η_X and η_Y, the elements $(X, \overline{\nu}_X)$ and $(Y, \overline{\nu}_Y)$ are equal in the bordism group $\Omega_4^{\mathrm{TOPSpin}}(K_n)$. First of all, since X and Y have isometric intersection forms, their signatures are equal, and so they are spin bordant over a point, that is they are equal in $\Omega_4^{\mathrm{TOPSpin}}$. It remains to show that they are equal in $\ker d^2_{4,0} \subseteq H_4(K_n; \mathbb{Z})$, or in other words that $(X \sqcup Y, \overline{\nu}_X \sqcup -\overline{\nu}_Y)$ determines the trivial element of $H_4(K_n; \mathbb{Z})$. The element in $H_4(K_n; \mathbb{Z})$ is given by $(\eta_X)_*[X] - (\eta_Y)_*[Y]$. To prove that an element of $z \in H_4(K_n; \mathbb{Z})$ vanishes, it suffices to show that $\langle v, z \rangle = 0$ for every $v \in H^4(K_n; \mathbb{Z}) \cong \mathrm{Hom}(H_4(K_n; \mathbb{Z}), \mathbb{Z})$. More precisely, it suffices to do this for v ranging over the generating set $x_i x_j$, for $1 \leq i \leq j \leq n$. We have, for each i and j:

$$\begin{aligned}
&\langle x_i x_j, (\eta_X)_*[X] - (\eta_Y)_*[Y] \rangle \\
=&\langle x_i x_j, (\eta_X)_*[X] \rangle - \langle x_i x_j, (\eta_Y)_*[Y] \rangle \\
=&\langle \eta_X^*(x_i) \eta_Y^*(x_j), [X] \rangle - \langle \eta_Y^*(x_i) \eta_Y^*(x_j), [Y] \rangle \\
=&\lambda(\mathrm{PD} \circ \eta_X^*(x_i), \mathrm{PD} \circ \eta_X^*(x_j)) - \lambda(\mathrm{PD} \circ \eta_Y^*(x_i), \mathrm{PD} \circ \eta_Y^*(x_j)) \\
=&\lambda(\mathrm{PD} \circ \eta_X^*(x_i), \mathrm{PD} \circ \eta_X^*(x_j)) - \lambda(F(\mathrm{PD} \circ \eta_X^*(x_i)), F(\mathrm{PD} \circ \eta_X^*(x_j))) \\
=&\lambda(\mathrm{PD} \circ \eta_X^*(x_i), \mathrm{PD} \circ \eta_X^*(x_j)) - \lambda(\mathrm{PD} \circ \eta_X^*(x_i), \mathrm{PD} \circ \eta_X^*(x_j)) \\
=&0.
\end{aligned}$$

Since this holds for each i, j, we deduce that $(\eta_X)_*[X] - (\eta_Y)_*[Y] = 0 \in H_4(K_n; \mathbb{Z})$ as desired. This completes the proof that $(X, \overline{\nu}_X)$ and $(Y, \overline{\nu}_Y)$ are bordant, and therefore completes the proof of the classification of spin 4-manifolds in terms of their intersection forms.

Remark 7.8. The proof in the non-spin case is broadly similar, but some details are different, in particular the normal 2-types are different, and the differentials in the spectral sequence have to take w_2 into account. I hope that even though it is just a proof for one case, it has nevertheless given an idea of the kinds of computations one needs to perform in order to classify 4-manifolds using modified surgery.

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