NON-SMOOTHABLE HOMEOMORPHISMS OF 4-MANIFOLDS WITH BOUNDARY

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Abstract. We construct the first examples of non-smoothable self-homeomorphisms of smooth 4-manifolds with boundary that fix the boundary and act trivially on homology. As a corollary, we construct self-diffeomorphisms of 4-manifolds with boundary that fix the boundary and act trivially on homology but cannot be isotoped to any self-diffeomorphism supported in a collar of the boundary and, in particular, are not isotopic to any generalised Dehn twist.

1. Introduction

1.1. Results.

Let X be a smooth, compact, oriented 4-manifold with boundary. We will denote by $\operatorname{Homeo}^+(X, \partial X)$ the topological group of orientation-preserving self-homeomorphisms of X that restrict to the identity map on ∂X, topologised using the compact-open topology. We say that a homeomorphism $f \in \operatorname{Homeo}^+(X, \partial X)$ is *non-smoothable* if it is not isotopic *relative to the boundary* to any self-diffeomorphism of X.

We denote by $\operatorname{Tor}(X, \partial X) \subset \pi_0 \operatorname{Homeo}^+(X, \partial X)$ the (*topological*) *Torelli group* of $(X, \partial X)$, the subgroup of isotopy classes of homeomorphisms that induce the identity map on $H_2(X)$. When X is simply-connected and *closed*, Perron–Quinn [Per86, Qui86] showed that $\operatorname{Tor}(X, \partial X)$ is trivial and hence all of its elements are smoothable. However, if $\partial X \neq \emptyset$ then Orson–Powell [OP23] showed that it is in general non-trivial and hence could contain non-smoothable homeomorphisms. Our first result shows the existence of such non-smoothable elements of $\operatorname{Tor}(X, \partial X)$.

Theorem 1.1. *There exists an infinite family of pairwise non-diffeomorphic compact, oriented, smooth, simply-connected 4-manifolds with connected boundary $\{(X_n, \partial X_n)\}_{n\in\mathbb{N}}$ and $\operatorname{Tor}(X_n, \partial X_n)$ of infinite order such that, for each n, all non-trivial elements in $\operatorname{Tor}(X_n, \partial X_n)$ are non-smoothable.*

In fact, we construct two separate such families, one such that the boundaries ∂X_n are pairwise non-diffeomorphic (Theorem 4.1) and another family $\{Z_n\}_{n\in\mathbb{N}}$ such that the boundaries ∂Z_n are all diffeomorphic and the Z_n are all homeomorphic relative to their boundaries (Theorem 4.4). Furthermore, the first of these families is minimal in the sense that the produced manifolds have the simplest possible intersection forms. See Remark 2.8 for more details.

We also note that the homeomorphisms of Theorem 1.1 are not isotopic to any diffeomorphism even 'absolutely', i.e. when considering isotopies that do not fix the boundary pointwise. Indeed, we will see in Section 5 that relative and absolute non-smoothability are equivalent notions for 4-dimensional manifolds.

It is easy to describe a class of smoothable maps in $\operatorname{Tor}(X, \partial X)$. Given a loop γ of orientation-preserving diffeomorphisms of the boundary based at the identity, we can form a diffeomorphism $\varphi_\gamma\colon (X, \partial X) \to (X, \partial X)$ by inserting γ into a collar of the boundary and extending via the identity map. Such diffeomorphisms are called *generalised Dehn twists*, and, since they are supported on a collar of the boundary, they represent (smoothable) elements in $\mathrm{Tor}(X, \partial X)$. It is an interesting question whether a given smoothable element of $\mathrm{Tor}(X, \partial X)$ is realised by a generalised Dehn twist.

Our second result shows the non-realisability of smoothable elements of $\mathrm{Tor}(X, \partial X)$ by generalised Dehn twists.

Theorem 1.2. *There exists an infinite family of pairwise non-diffeomorphic compact, oriented, smooth, simply-connected 4-manifolds with connected boundary* $\{(W_n, \partial W_n)\}_{n\in\mathbb{N}}$ *and* $\mathrm{Tor}(W_n, \partial W_n)$ *of infinite order, such that all mapping classes in* $\mathrm{Tor}(W_n, \partial W_n)$ *are smoothable, but only the identity map is supported on a collar of the boundary and, in particular, only the identity map is realised by a generalised Dehn twist.*

1.2. Background.

The question about smoothable versus non-smoothable homeomorphisms for closed, oriented, simply-connected 4-manifolds has been studied extensively. If X is such a manifold (or has boundary a homology sphere) with indefinite intersection form or the rank of $H_2(X)$ at most 8, then Wall [Wal64] showed that all isometries of the intersection form $\mathrm{Aut}(H_2(X\#(\mathbb{S}^2\times\mathbb{S}^2)), \lambda_{X\#(\mathbb{S}^2\times\mathbb{S}^2)})$ can be realised by diffeomorphisms (and hence all self-homeomorphisms of $X\#(\mathbb{S}^2\times\mathbb{S}^2)$ are smoothable by Perron–Quinn [Per86, Qui86]). Ruberman and Strle [RS23] extended this result to show that any self-homeomorphism of $X\#(\mathbb{S}^2\times\mathbb{S}^2)$ that acts trivially on the homology of the $\mathbb{S}^2\times\mathbb{S}^2$-summand is smoothable.

Conversely, for closed 4-manifolds, Friedman and Morgan constructed the first examples of non-smoothable homeomorphisms by considering self-homeomorphisms of Dolgachev surfaces [FM88]. The general argument for producing such non-smoothable homeomorphisms goes in the following manner. Firstly, by Freedman [Fre82], we know that any automorphism of the intersection form is realisable by a homeomorphism. Then one uses a gauge-theoretic invariant (e.g. Seiberg-Witten invariants) to show that certain automorphisms of the intersection form are not realisable by a diffeomorphism, since diffeomorphisms must preserve certain homology classes (e.g. Seiberg-Witten basic classes). This style of argument has been used to produce many more examples. In particular, Donaldson [Don90] showed that the $K3$ surface admits a non-smoothable homeomorphism. Further instances are known, see [MS97], [Bar21].

There is a natural generalisation of this idea to the case of 4-manifolds with boundary using Monopole or Heegaard-Floer homology [KM07][OS06] which consists of looking at the cobordism maps induced in Floer homology by the 4-manifold together with a spin^c-structure. Indeed, two spin^c-structures related by a diffeomorphism fixing the boundary induce the same cobordism map up to multiplication by ± 1, depending on the action of the diffeomorphism on homology. However, this approach cannot obstruct the smoothability of homeomorphisms in the Torelli group because, as we will see in Section 2, such homeomorphisms preserve the relative isomorphism class of a spin^c-structure since they act trivially on $H_2(X, \partial X)$.

A different approach, based on Seiberg-Witten Floer stable homotopy type [Man03], has been used recently by Konno and Taniguchi [KT22, Thm 1.7] to construct non-smoothable homeomorphisms for a large class of 4-manifolds with boundary a rational homology sphere. The technical results underpinning this approach require $b_1(\partial X) = 0$ and therefore this approach cannot be directly applied to find non-smoothable elements in the Torelli group because the latter is non-trivial only when $b_1(\partial X) \geq 2$ (see Theorem 2.7). An enhancement of this approach to the case $b_1(\partial X) > 0$ might become available in the future using generalisations of [Man03], e.g. [KLS18],[SS21]. Regardless, in this paper we take a different approach based on embedding X into a closed 4-manifold (see Section 2.3).

1.3. **Outline.** We briefly outline the contents of the paper. In Section 2 we recall the classification of $\mathrm{Tor}(X, \partial X)$ in terms of algebraic objects called variations, and prove a key lemma (Lemma 2.9) which we will use to detect elements of the Torelli group. In Section 3 we prove technical conditions under which we can guarantee the existence of non-smoothable elements of the Torelli group. In Section 4 we use the conditions from the previous section to produce our two infinite families of examples and hence prove Theorem 1.1. Finally, in Section 5 we consider generalised Dehn twists and prove Theorem 1.2.

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2. VARIATIONS

2.1. **Definitions.** The aim of this section is to describe the classification of homeomorphisms up to isotopy for simply-connected, topological, oriented 4-manifolds with boundary. This classification is due to the work of Osamu Saeki, Patrick Orson, and Mark Powell [Sae06, OP23]. We will begin by defining what a variation is, which is the central object involved in the classification. Unlike the rest of this paper, all of the statements in this section are purely topological in nature, and so hold regardless of whether the manifolds in question are smooth or topological.

Definition 2.1. Let X be a simply-connected, oriented 4-manifold with boundary and let $f \in \mathrm{Homeo}^+(X, \partial X)$ be an orientation-preserving homeomorphism relative to the boundary ∂X. Then the *variation induced by* f, denoted as Δ_f, is defined as

$$\begin{aligned} \Delta_f \colon H_2(X, \partial X) &\to H_2(X) \\ [\Sigma] &\mapsto [\Sigma - f(\Sigma)], \end{aligned}$$

where Σ denotes a relative 2-chain. Note that the homology class $[\Sigma - f(\Sigma)]$ does not depend on the choice of representative relative 2-chain Σ [OP23, Sec. 2.2].

We can also define variations without reference to a homeomorphism.

Definition 2.2. Let X be a simply-connected, oriented 4-manifold with boundary and let $\Delta \colon H_2(X, \partial X) \to H_2(X)$ be a homomorphism. Then we say that Δ is a *Poincaré variation* if

$$\Delta + \Delta^! = \Delta \circ j_* \circ \Delta^! \colon H_2(X, \partial X) \to H_2(X),$$

where j is the inclusion map of pairs $(X, \emptyset) \to (X, \partial X)$ and $\Delta^!$ denotes the 'umkehr' homomorphism to Δ, defined as the following composition:

$$\Delta^! \colon H_2(X, \partial X) \xrightarrow{\mathrm{PD}^{-1}} H^2(X) \xrightarrow{\mathrm{ev}} H_2(X)^* \xrightarrow{\Delta^*} H_2(X, \partial X)^* \xrightarrow{\mathrm{ev}^{-1}} H^2(X, \partial X) \xrightarrow{\mathrm{PD}} H_2(X).$$

Following the notation of Orson-Powell, we will denote the set of Poincaré variations of $(X, \partial X)$ as $\mathcal{V}(H_2(X), \lambda_X)$, where λ_X denotes the intersection form of X. This notation is used because it is shown in [OP23, Sec. 7] that the set of variations only depends on the isometry class of the intersection form $(H_2(X), \lambda_X)$, rather than on the 4-manifold specifically.

We can give $\mathcal{V}(H_2(X), \lambda_X)$ the structure of a group due to the following lemma of Saeki.

Lemma 2.3 ([Sae06, Lem. 3.5])**.** *The set $\mathcal{V}(H_2(X), \lambda_X)$ forms a group with multiplication given by*

$$\Delta_1 \cdot \Delta_2 := \Delta_1 + (\mathrm{Id} - \Delta_1 \circ j_*) \circ \Delta_2,$$

identity the zero homomorphism, and inverse given by

$$\Delta^{-1} = -(\mathrm{Id} - \Delta \circ j_*) \circ \Delta.$$

Further, we have

Lemma 2.4 ([Sae06, Lem. 3.2])**.** *Let X be a compact, simply-connected, oriented, topological 4-manifold with boundary ∂X and let $f \in \mathrm{Homeo}^+(X, \partial X)$. Then Δ_f is a Poincaré variation.*

The converse of the above result, that all Poincaré variations are induced via homeomorphisms, is given by [OP23, Thm. A].

2.2. **The Torelli group.** The map which sends a homeomorphism to its variation gives a factorisation of the map which takes the induced automorphism of the form for a homeomorphism:

$$\pi_0 \mathrm{Homeo}^+(X, \partial X) \xrightarrow{f \mapsto \Delta_f} \mathcal{V}(H_2(X), \lambda_X) \xrightarrow{\Delta \mapsto \mathrm{Id} - \Delta \circ q} \mathrm{Aut}(H_2(X), \lambda_X), \tag{2.1}$$

where $q \colon H_2(X) \to H_2(X, \partial X)$ is the quotient map. It is the result of Freedman–Perron–Quinn [Fre82, Per86, Qui86] that, for a *closed*, simply-connected 4-manifold X, the above composition is a bijection. Hence, all homeomorphisms of a closed simply-connected 4-manifold that map to the trivial element of $\mathrm{Aut}(H_2(X), \lambda_X)$ are isotopic to the identity map. For manifolds with non-empty boundary, the classification is more subtle [OP23, Thm. A], and in particular we can have homeomorphisms that are not isotopic to the identity but still induce the trivial element of $\mathrm{Aut}(H_2(X), \lambda_X)$ under 2.1.

Definition 2.5. Let X be a compact, simply-connected, oriented, 4-manifold with boundary ∂X. We define the *Torelli group* $\mathrm{Tor}(X, \partial X) \subset \pi_0 \mathrm{Homeo}^+(X, \partial X)$ to be the subgroup of homeomorphisms that induce the trivial element of $\mathrm{Aut}(H_2(X), \lambda_X)$ under 2.1.

Note that the subgroup of variations which are induced by elements in the Torelli group is exactly the subgroup of variations satisfying that $\Delta \circ q \colon H_2(X) \to H_2(X)$ is the zero map. For such Poincaré variations, we can construct a skew-symmetric pairing in the following way. Let Δ be a Poincaré variation. Then this gives rise to a map

$$(\eta_\Delta)^{\mathrm{ad}} \colon H_1(\partial X) \to H_2(\partial X) \cong H_1(\partial X)^*$$

(note that the last isomorphism is given by Poincaré duality and universal coefficients) by first lifting an element in $H_1(\partial X)$ to an element in $H_2(X, \partial X)$, mapping to $H_2(X)$ using Δ (note that this image does not depend on the choice of lift) and then noting that by the definition of Poincaré variations, this element lifts uniquely to an element in $H_2(\partial X)$. As suggested by the notation, we can interpret this map as the adjoint of a pairing:

$$\eta_\Delta \colon H_1(\partial X) \times H_1(\partial X) \to \mathbb{Z}$$

and it is stated in [Sae06, Prop. 4.2] that this form is skew-symmetric. To see this, it is enough to verify that $\eta_\Delta^{\mathrm{ad}}(x)(x) = 0$ for any $x \in H_1(\partial X)$, and this fact is geometrically clear from the definition of η^{ad}. More crucially, we can go the other way. Let $\eta \colon H_1(\partial X) \times H_1(\partial X) \to \mathbb{Z}$ be a skew-symmetric pairing. Then we can define a variation Δ_η as the following composition:

$$H_2(X, \partial X) \xrightarrow{\partial} H_1(\partial X) \xrightarrow{\eta^{\mathrm{ad}}} H_1(\partial X)^* \xrightarrow{\mathrm{ev}^{-1}} H^1(X) \xrightarrow{PD} H_2(\partial X) \xrightarrow{i_*} H_2(X) \tag{2.2}$$

where the map ∂ denotes the connecting homomorphism in the long exact sequence of the pair and $i\colon \partial X \to X$ denotes the inclusion. We have the following sequence, due to Saeki.

Proposition 2.6 ([Sae06, Prop. 4.2],[OP23, Thm. 7.13])**.** *Let X be a compact, simply-connected, oriented, topological 4-manifold with* connected *boundary ∂X. Then the following is a short exact sequence:*

$$0 \to \Lambda^2 H_1(\partial X)^* \to \mathcal{V}(H_2(X), \lambda_X) \to \operatorname{Aut}(H_2(X), \lambda_X) \to 0.$$

So it follows that, in the connected boundary case, we have that the variations which induce the trivial map on homology are in one-to-one correspondence with skew-symmetric forms on $H_1(\partial X)$.

We have the following classification of the Torelli group, due to Orson-Powell.

Theorem 2.7 ([OP23, Cor. D])**.** *Let X be a compact, simply-connected, oriented 4-manifold with connected boundary ∂X. Then there is an isomorphism of groups:*

$$\operatorname{Tor}(X, \partial X) \xrightarrow{\cong} \Lambda^2 H_1(\partial X)^*,$$
$$[f] \mapsto \eta_{\Delta_f}.$$

Remark 2.8. It follows from this that $\operatorname{Tor}(X, \partial X)$ is non-trivial if and only if $b_1(\partial X) \geq 2$. In fact, we can say more. Since X is simply-connected, it must also have $b_2(X) \geq b_1(\partial X)$ and, if $b_2(X) = b_1(\partial X)$, vanishing intersection form. This follows from the exact sequence

$$0 \to H_2(\partial X) \to H_2(X) \xrightarrow{\lambda^{\mathrm{ad}}} H_2(X)^* \to H_1(\partial X) \to 0,$$

where λ^{ad} is the adjoint of the intersection form and the penultimate map is the composition of the inverse of the evaluation map, Poincaré duality and the connecting morphism of the long exact sequence of the pair. After tensoring with $\mathbb{Q}$ and using that $b_2(X) = b_1(\partial X)$, the claim is clear. It follows that examples of 4-manifolds with $\operatorname{Tor}(X, \partial X)$ non-trivial must have $b_2(X) \geq 2$.

2.3. Applying variations to closed manifolds.

Let W be a simply-connected, oriented manifold with boundary ∂W. In Section 3 we will want to use variations to prove that a homeomorphism $f\colon (W, \partial W) \to (W, \partial W)$ is non-smoothable. In doing so, we will need the following lemma.

Lemma 2.9. *Let W_1 be a simply-connected, oriented 4-manifold with boundary $\partial W_1 \cong Y$, W_2 be an oriented 4-manifold with boundary $\partial W_2 \cong -Y$ and let $X := W_1 \cup_Y W_2$ be the closed, oriented union. Let $\eta\colon H_1(\partial W_1) \times H_1(\partial W_1) \to \mathbb{Z}$ be a skew-symmetric pairing, denote by Δ_η the induced variation (given by Equation (2.2)) and denote by $\varphi_\eta\colon W_1 \to W_1$ the induced homeomorphism (given by Theorem 2.7). Consider the umkehr map to the inclusion $i_1\colon W_1 \to X$,*

$$i_1^!\colon H_2(X) \xrightarrow{\mathrm{PD}^{-1}} H^2(X) \xrightarrow{i_1^*} H^2(W_1) \xrightarrow{\mathrm{PD}} H_2(W_1, Y).$$

Then for any class $x \in H_2(X)$ we have that

$$(\varphi_\eta \cup \operatorname{Id}_{W_2})_*(x) = x - (i_1)_* \Delta_\eta(i_1^!(x)) \in H_2(X), \tag{2.3}$$

where $\varphi_\eta \cup \operatorname{Id}_{W_2}\colon X \to X$ is the homeomorphism defined as φ_η on W_1 and as Id_{W_2} on W_2.

Proof. Let $\Sigma \subset X$ be an embedded, closed, oriented surface representing x, transverse to Y (for topological transversality see [FQ90, Thm. 9.5A]).

The statement $i_1^!(x) = [\Sigma \cap W_1] \in H_2(W_1, Y)$ is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
H_2(X) \arrow[r, "q_*"] & H_2(X, W_2) & H_2(W_1, Y) \arrow[l, "\cong"'] \\
H^2(X) \arrow[u, "\text{PD}", "\cong"'] \arrow[r, "i_1^*"] & H^2(W_1) \arrow[ur, "\cong", "\text{PD}"'] &
\end{tikzcd}$$

where q_* is the map induced by the inclusion $q\colon (X, \emptyset) \to (X, W_2)$ which sends $[\Sigma]$ to $[\Sigma \cap W_1]$ and the written isomorphism $H_2(W_1, Y) \to H_2(X, W_2)$ comes from excision.

We now prove the commutativity of the diagram. Let $y \in H^2(X)$. Going first to the right, this maps to $i_1^*(y) \frown [W_1, Y] \in H_2(W_1, Y)$ and then to $y \frown (i_1)_*[W_1, Y] \in H_2(X, W_2)$ by naturality of the (relative) cap product. Going the other way, $y \in H^2(X)$ is mapped to $q_*(y \frown [X]) = y \frown q_*[X] \in H_2(X, W_2)$, again by naturality of the (relative) cap product. It remains to be shown that these are equal, i.e. that $q_*[X] = (i_1)_*[W_1, Y]$, but this is clear by the definition of q_*. This completes the proof that the diagram commutes.

As a singular 2-chain, $\Sigma = (\Sigma \cap W_1) + (\Sigma \cap W_2) \in C_2(X)$. Similarly, the singular 2-chain induced by $(\varphi_\eta \cup \mathrm{Id}_{W_2})(\Sigma)$, i.e. the left hand side of 2.3, is equal to the sum $\varphi_\eta(\Sigma \cap W_1) + (\Sigma \cap W_2)$ in $C_2(X)$. Hence we have that:

$$\Sigma - (\varphi_\eta \cup \mathrm{Id}_{W_2})(\Sigma) = (\Sigma \cap W_1) - \varphi_\eta(\Sigma \cap W_1) \in C_2(X).$$

The right hand side is homologous to the cycle induced by the glued-up surface $(\Sigma \cap W_1) \cup -\varphi_\eta(\Sigma \cap W_1)$ which is equal to $(i_1)_*\Delta_\eta([\Sigma \cap W_1])$ by Definition 2.1. □

3. Sufficient conditions for non-smoothability

In this section, all manifolds will be considered to be smooth. For any closed, oriented, 4-manifold X, we will denote by $\mathrm{Spin}^c(X)$ the set of isomorphism classes of spin^c-structures on X, and by $\mathcal{I}(X, \cdot)\colon \mathrm{Spin}^c(X) \to \mathcal{Y}$ a map taking values in an abelian group $\mathcal{Y}$ such that the action of $\mathrm{Diffeo}^+(X)$ on $H_2(X)$ by pull-back preserves the set of $\mathcal{I}$-basic classes, defined as

$$\mathcal{B}_{\mathcal{I}}(X) := \{c_1(\mathfrak{s}) \in H^2(X) \mid \mathcal{I}(X, \mathfrak{s}) \neq 0, \mathfrak{s} \in \mathrm{Spin}^c(X)\},$$

and moreover this set is *finite*. For example, if $b_2^+(X) \geq 2$, and $\mathfrak{s} \in \mathrm{Spin}^c(X)$, $\mathcal{I}(X, \mathfrak{s})$ may be taken to be the Seiberg-Witten invariant $SW(X, \mathfrak{s})$ [Wit94], the Ozsváth-Szabó mixed invariant $\Phi_{X,\mathfrak{s}}$ [OS06], or the Bauer-Furuta invariant $BF_{X,\mathfrak{s}}$ [BF04]. The finiteness of BF-basic classes is not stated explicitly in [BF04], but can be proved using curvature inequalities as observed in the proof of [MMP20, Thm. 4.5].

We now prove the main lemma that we will use to detect non-smoothability for homeomorphisms in the Torelli group.

Lemma 3.1. *Let W^4 be a compact, oriented 4-manifold with connected boundary Y. Suppose that $\pi_1(W) = 1$ and that $b_1(Y) \geq 2$. If W embeds in a* closed, *oriented 4-manifold X such that for some $\mathfrak{s} \in \mathrm{Spin}^c(X)$,*

(1) $\mathcal{I}(X, \mathfrak{s}) \neq 0$,
(2) $i_{Y,X}^*(c_1(\mathfrak{s})) \in H^2(Y)$ *is non-torsion where* $i_{Y,X} : Y \hookrightarrow X$ *is the inclusion,*
(3) $H^1(X \setminus W) = 0$,

then there exists infinitely many non-smoothable mapping classes in $\mathrm{Tor}(W, Y)$*. If in addition $b_1(Y) = 2$ then any non-trivial element of* $\mathrm{Tor}(W, Y)$ *is non-smoothable.*

Proof. To avoid clutter, it is convenient to denote $\zeta_X := c_1(\mathfrak{s})$ and its restrictions by $\zeta_W := i^*_{W,X}c_1(\mathfrak{s})$ and $\zeta_Y := i^*_{Y,X}c_1(\mathfrak{s})$.

Since $\zeta_Y \in H^2(Y;\mathbb{Z})$ is not torsion $\mathrm{PD}(\zeta_Y) = dv_1$, for some $d \in \mathbb{Z}\setminus\{0\}$ and an indivisible element $v_1 \in H_1(Y;\mathbb{Z})$. Extend v_1 to $v_1,\dots,v_{b_1(Y)} \in H_1(Y)$, a lift of a basis of $H_1(Y)/\mathrm{Torsion}(H_1(Y))$. Now we set $\eta := v_1^* \wedge v_2^* \in \Lambda^2(H_1(Y)^*)$ where v_i^* denotes the Hom dual with respect to the above basis (note that $v_2 \neq 0$ exists since we assumed $b_1(Y) \geq 2$).

By Theorem 2.7, for each $k \in \mathbb{Z}\setminus\{0\}$, there is a unique mapping class in $\mathrm{Tor}(W,Y)$ associated to $k\eta$, and we define $\varphi_k \in \mathrm{Homeo}^+(W,Y)$ to be an arbitrary representative of that class. By construction, each φ_k acts trivially on $H_2(W)$. The rest of the proof is devoted to showing that φ_k is non-smoothable for infinitely many values of k.

For each k, we define $\hat{\varphi}_k := \varphi_k \cup \mathrm{Id}_{X\setminus\mathrm{int}(W)} \in \mathrm{Homeo}^+(X)$ as in Lemma 2.9 to be the homeomorphism obtained by extending φ_k as the identity on $X\setminus W$. The non-smoothability of $\hat{\varphi}_k$ for infinitely many k, which we are now going to prove, implies the analogous statement for φ_k.

We will show that $\{\hat{\varphi}_k^*\zeta_X\}_{k\in\mathbb{Z}\setminus\{0\}}$ is infinite. Since $\mathcal{B}_{\mathcal{I}}(X)$ is finite, and is preserved by the action of $\mathrm{Diff}^+(X)$, we will reach a contradiction.

It follows from Lemma 2.9 that

$$(\hat{\varphi}_k)_*(\mathrm{PD}(\zeta_X)) = \mathrm{PD}(\zeta_X) - (i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) \in H_2(X). \tag{3.1}$$

From (2.2) we see that

$$(i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) = k\cdot(i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \partial \circ \mathrm{PD}(\zeta_W) \in H_2(X).$$

We claim that the right hand side is equal to a non-torsion element times k. This will imply the desired result by (3.1). We have that

$$\begin{aligned}(i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) &= k\cdot(i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \partial \circ \mathrm{PD}(\zeta_W)\\ &= k\cdot(i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \mathrm{PD}(\zeta_Y)\\ &= k\cdot(i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad}(dv_1)\\ &= kd\cdot(i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1}(v_2^*).\end{aligned}$$

Since v_2 is non-torsion and the maps ev^{-1} and PD are isomorphisms, the claim will follow if we prove that $(i_{Y,X})_*\colon H_2(Y) \to H_2(X)$ is injective. Now $(i_{Y,X})_* = (i_{W,X})_* \circ (i_{Y,W})_*$. The kernel of $(i_{Y,W})_*$ is equal to the image of $H_3(W,Y) \to H_2(Y)$ in the long exact sequence of the pair, which is trivial since $H_3(W,Y) \cong H^1(W) = 0$. Similarly the kernel of $(i_{W,X})_*$ is equal to the image of $H_3(X,W) \to H_2(W)$ which is zero because $H_3(X,W) \cong H_3(X\setminus\mathrm{int}(W),Y)$ by excision and by assumption $0 = H^1(X\setminus W) \cong H_3(X\setminus\mathrm{int}(W),Y)$. Being the composition of injective maps, $i_{Y,W}$ is injective. This completes the proof that there are infinitely many non-smoothable mapping classes in $\mathrm{Tor}(W,Y)$.

To prove the last statement we assume now that $b_1(Y) = 2$. Then, under the isomorphism from Theorem 2.7, we can identify $\mathrm{Tor}(W,Y)$ with the infinite cyclic group generated by η. Above we showed that there exists $k_0 > 0$ such that $\varphi_{k\eta}$ is non-smoothable for any $|k| > k_0$. The non-smoothability of $\mathrm{Tor}(X,Y)\setminus\{\mathrm{Id}_X\}$ follows from this by using the fact that smoothable mapping classes form a subgroup of $\mathrm{Tor}(X,Y)$ and that all non-trivial subgroups of $\mathbb{Z}$ are infinite. □

Lemma 3.1 has an immediate application to symplectic fillings. For the reader's convenience we recall that a strong symplectic filling of a contact 3-manifold (Y,ξ) is a compact, symplectic

4-manifold (W, ω) with oriented boundary Y such that such that there exists a Liouville vector field V defined in a neighbourhood of ∂W pointing outwards along Y and such that the pull-back of the 1-form $\omega(V, \cdot)$ to Y induces the contact structure ξ on Y. The interested reader is referred to [Gei08, OS04].

Corollary 3.2. *Let (W, ω) be a strong symplectic filling of (Y, ξ). Further suppose that W is simply-connected, $b_1(Y) \geq 2$ and that $c_1(\xi) \in H^2(Y)$ is not torsion. Then the same conclusions of Lemma 3.1 hold.*

Proof. It is possible to embed (W, ω) symplectically into a closed symplectic 4-manifold (X, ω_X) [Eli04, Etn04] (see also [LM97] for the Stein case). Furthermore, we can arrange that $\pi_1(X \setminus W) = 1$ and that $b_2^+(X) \geq 2$ [EMM22, Sec. 6] (note that the symplectic cap called X in [EMM22, Sec. 6] plays the role of $X \setminus \operatorname{int}(W)$ in our proof). Being symplectic, it follows from Taubes' work [Tau94] that $c_1(X)$ is a Seiberg-Witten basic class. Because the embedding is symplectic, we have that $i^*_{Y,X} c_1(X) = i^*_{Y,W} \circ i^*_{W,X} c_1(X) = c_1(\xi)$, which is non-torsion by assumption. Now apply Lemma 3.1. □

4. Constructing examples

In this section we will construct two infinite families of 4-manifolds, each supporting an infinite family of non-smoothable mapping classes in their Torelli groups.

4.1. **Family from Legendrian surgery.** Stein domains (see [GS99, Sec. 11.2] for an introduction) are a particular case of symplectic manifolds which are also strong symplectic fillings of their boundary [Gei08, Prop. 5.4.9]. A 4-manifold can be given the structure of a Stein domain if and only if it can be described by a special type of Kirby diagram [Gom98]: a *Legendrian link diagram in standard form* [GS99, Def. 11.1.7] where the framing coefficient on each link component K is equal to $\operatorname{tb}(K) - 1$, where tb denotes the Thurston-Bennequin invariant. Given such a diagram, we can easily compute the first Chern class of the Stein domain as follows [Gom98, Prop. 2.3]:

$$c_1(\xi) = \left[\sum_{i=1}^{N} \operatorname{rot}(K_i) h^*_{K_i} \right] \in H^2(W), \tag{4.1}$$

where W is the Stein domain specified by the diagram, $K_1, \ldots, K_N$ are the *oriented* Legendrian components in the diagram, $h^*_{K_i} \in C^2(W)$ denotes the cochain associated to the 2-handle attached along K_i and $\operatorname{rot}(K_i)$ is the rotation number of the component K_i.

We are now ready to define our first family of manifolds. For any $n \in \mathbb{N}$, we define X_n to be the Stein domain specified by the two component Legendrian link diagram in standard form in Figure 1.

Theorem 4.1. *For each $n \in \mathbb{N}$ the 4-manifold X_n is simply-connected, has $H_2(X) \cong \mathbb{Z}^2$ and vanishing intersection form. Moreover, all the non-trivial elements of the topological Torelli group $\operatorname{Tor}(X_n, \partial X_n)$ are non-smoothable. Furthermore, define $n_r := 5 \cdot 2^{r-1} - 3$ for $r \geq 1$. Then ∂X_{n_r} is not diffeomorphic to ∂X_{n_m} for any $r \neq m$.*

Proof. The first part follows from the fact that X_n is a link trace on a link with two components which are both 0-framed and with vanishing linking number. From this it also follows that $H_1(\partial X_n) \cong \mathbb{Z}^2$.

We want to invoke Corollary 3.2. By construction X_n is a Stein domain, hence it only remains to show that $c_1(\xi_n) \neq 0$, with ξ_n being the contact structure induced on ∂X_n.

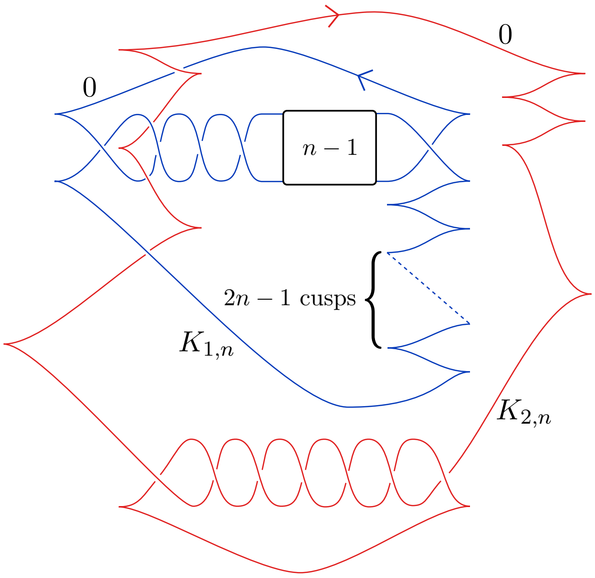

FIGURE 1. A Legendrian link diagram in standard form for X_n. The knot $K_{1,n}$ with the specified orientation has $2n+3$ crossings, $2n+2$ right cusps and rotation number $2n$. The knot $K_{2,n}$ has rotation number 0.

Denote by $x_{1,n}, x_{2,n} \in H_2(X_n)$ the homology classes induced by the 2-handles attached along $K_{1,n}$ and $K_{2,n}$, respectively.

By (4.1), we have that:

$$c_1(X_n) = \operatorname{rot}(K_{1,n})x^*_{1,n} + \operatorname{rot}(K_{2,n})x^*_{2,n} = 2nx^*_{1,n},$$

with respect to the dual basis $x^*_{1,n}, x^*_{2,n} \in H^2(X_n)$. The first Chern class of the Stein structure restricts to that of the contact structure on the boundary and the inclusion map $i_{\partial X_n} \colon \partial X_n \hookrightarrow X_n$ induces an isomorphism $H^2(X_n) \xrightarrow{\cong} H^2(\partial X_n)$, as can be seen from the long exact sequence of the pair and using the fact that the intersection form of X_n is trivial and $H_1(X_n) = 0$. It follows immediately that Corollary 3.2 applies.

It remains to show that $\partial X_{n_r} \neq \partial X_{n_m}$ for $r \neq m$. In the rest of the proof we will identify $H_2(X_n) \cong \mathbb{Z}^2$ via $x_{i,n} \mapsto e_i$, $i = 1, 2$, and then identify $H_2(\partial X_n) \cong \mathbb{Z}^2$ by composing the previous identification with the isomorphism $H_2(\partial X_n) \xrightarrow{\cong} H_2(X_n)$ given by the inclusion.

For $A \in GL(\mathbb{Z}, 2)$, we define

$$\mathcal{G}_{X_n}(A) := \max\{g_{X_n}(A \cdot e_1), g_{X_n}(A \cdot e_2)\},$$

where $g_{X_n} : H_2(X_n)/\operatorname{Torsion}(H_2(X_n)) \to \mathbb{Z}_{\geq 0}$ is the minimal genus function [GS99, p. 37].

The adjunction inequality for Stein domains [LM97] (see also [Akb16, Prop. 9.2] or [GS99, Thm. 11.4.7]) gives:

$$\mathcal{G}_{X_n}(A) \geq 1 + \frac{1}{2}\max_{i=1,2} |\langle c_1(X), A \cdot e_i\rangle| \geq 1 + 2n\max\{|A_{11}|, |A_{21}|\} \geq 1 + 2n \tag{4.2}$$

where in the last inequality we have used that $\det(A) \neq 0$.

Now suppose for a contradiction that $f \colon \partial X_{n_r} \to \partial X_{n_m}$ is a diffeomorphism. By swapping to f^{-1} if necessary, we can suppose that $r < m$.

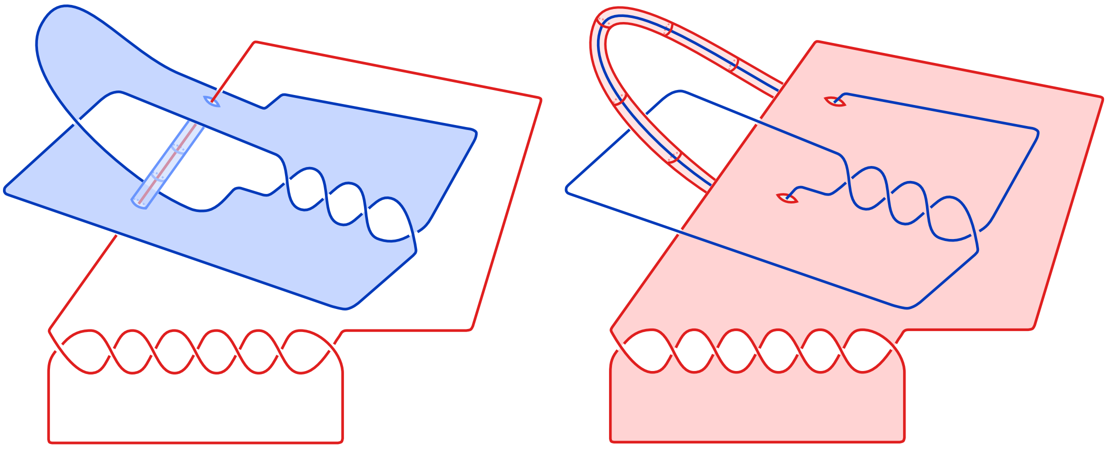

FIGURE 2. The pictures show, in a surgery presentation for ∂X_n (every link component is 0-framed), the two surfaces $\Sigma_{1,n}$ (on the left) and $\Sigma_{2,n}$ (on the right) for the case $n = 1$. They have genera $g(\Sigma_{1,n}) = 2n + 3$ and $g(\Sigma_{2,n}) = 7$.

Under the identifications introduced above, $f_*\colon H_2(\partial X_{n_r}) \to H_2(\partial X_{n_m})$ induces an element $A \in GL(\mathbb{Z}, 2)$.

We can find closed, orientable surfaces $\Sigma_{i,n_r} \subset \partial X_{n_r}$ representing $e_i \in H_2(\partial X_{n_r})$, for $i = 1, 2$ of genera $g(\Sigma_{1,n_r}) = 2n_r + 3$ and $g(\Sigma_{2,n_r}) = 7$; see Figure 2. Hence $f(\Sigma_{i,n_r}) \subset \partial X_{n_m}$ provides us with the upper bound $\mathcal{G}_{X_{n_m}}(A) \leq \max\{2n_r + 3, 7\} \leq 2n_r + 3$.

This together with (4.2) gives

$$5 \cdot 2^r - 3 = 2n_r + 3 \geq \mathcal{G}_{X_{n_m}}(A) \geq 1 + 2n_m = 5 \cdot 2^m - 5,$$

which is impossible for $r < m$. □

Remark 4.2. By mimicking the construction of the manifolds X_n, it is not difficult to construct another family of manifolds $\{Q_n\}_{n\in\mathbb{N}}$ with $b_1(\partial Q_n) \to \infty$ as $n \to \infty$ and with $\mathrm{Tor}(Q_n, \partial Q_n)$ containing infinitely many non-smoothable mapping classes. For example, one can define Q_n recursively: start with $Q_1 = X_1$ and obtain Q_{n+1} from Q_n by attaching a 2-handle along a knot C_{n+1}, where $C_1 = K_{2,1}$ and, for $n \geq 1$, C_{n+1} is a knot identical to $K_{2,1}$, but unlinked from all components besides C_n, and linking C_n in the same way that $K_{2,1}$ links $K_{1,1}$. Then Q_n will have a Stein structure satisfying the hypothesis of Corollary 3.2 and $b_1(\partial Q_n) = 1 + n$.

4.2. **Family from knot surgery.** Now we will construct an infinite family of manifolds sharing the same boundary. It is possible to give a proof of Theorem 4.4 below by pairing Corollary 3.2 together with well known compactification results for Stein domains [Eli04, Etn04, EMM22, LM97] but we decided to use a more elementary approach here which does not rely on symplectic topology.

Let Z be the 4-manifold with boundary defined by the Kirby diagram in Figure 3 (a). Let $T \subset \mathrm{int}(Z)$ be the embedded torus obtained by capping the genus one Seifert surface for the red trefoil knot with the core of the handle attached along it. From the diagram it is clear that $[T] \neq 0 \in H_2(Z)$ and $[T]^2 = 0$.

For any $n \in \mathbb{N}$ we define the knot $K(n)$ to be the twist knot with Alexander polynomial $\Delta_{K(n)} = -(2n-1) + n(t + t^{-1})$, and $E_{K(n)}$ to be the knot exterior of $K(n)$ in S^3. Then we define $Z_n := E(K) \times S^1 \cup_\partial (Z \setminus \nu T)$ to be the manifold obtained by performing knot surgery

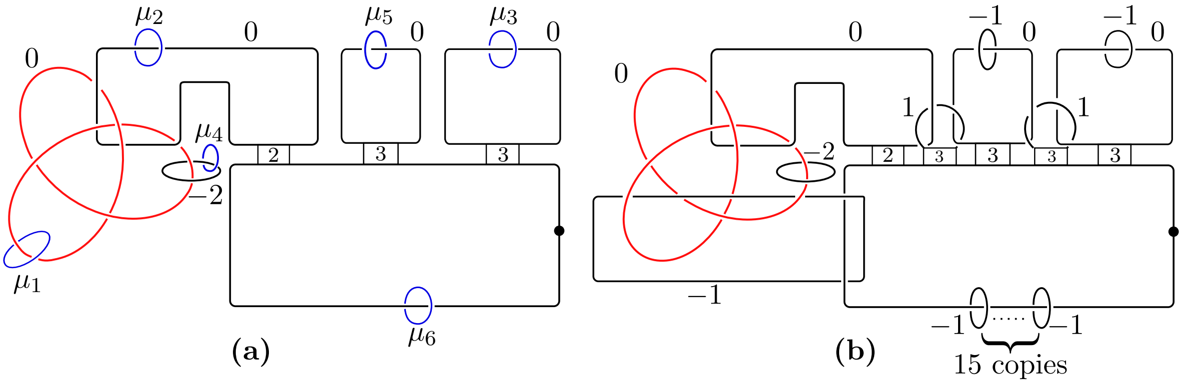

Figure 3. (a) Kirby diagram for the 4-manifold Z, (b) Kirby diagram showing an embedding of Z into $K_3\#2\overline{\mathbb{CP}}^2$.

[FS98] on the torus T using the knot $K(n)$. Since the knot surgery only changes the manifold in the interior, we have an identification $\partial Z_n \cong Y := \partial Z$.

Proposition 4.3. *The 4-manifolds $\{Z_n\}_{n\in\mathbb{N}}$ are all homeomorphic to Z relative to Y.*

Proof. In [FS98] it is shown that knot surgery preserves the homeomorphism type of a closed, simply-connected manifold provided that the surgered torus has simply-connected complement, but an additional argument is required when the manifold has non-empty boundary. The torus $T \subset Z$ embeds in a Gompf nucleus $N(2)$ [Gom91] by construction and so, in particular, it has simply-connected complement. This implies that all of the Z_n are simply-connected by a routine Seifert-Van Kampen argument. Boyer [Boy86, Thm. 0.7] tells us that there are two obstructions to extending the homeomorphism $\mathrm{Id}\colon Y \to Y$ to a homeomorphism $f\colon Z_n \to Z$. The first is to find a 'morphism', namely an isometry Λ of the intersection forms such that

$$\begin{tikzcd}
H_2(Y) \arrow[r] \arrow[d, equal] & H_2(Z_n) \arrow[r] \arrow[d, "\Lambda"] & H_2(Z_n)^* \arrow[r] & H_1(Y) \arrow[d, equal] \\
H_2(Y) \arrow[r] & H_2(Z) \arrow[r] & H_2(Z)^* \arrow[r] \arrow[u, "\Lambda^*"] & H_1(Y)
\end{tikzcd}$$

commutes, where the rows in the above diagram are the same rows used in Remark 2.8.

We can find such a Λ in the following way. Define a map $g\colon Z_n \to Z$ as the identity on the complement of T and as the standard degree one map from $E_{K(n)} \to \mathbb{D}^2 \times \mathbb{S}^1$ times the identity map on the $\mathbb{S}^1$-factor on $E_{K(n)} \times \mathbb{S}^1$. Then $\Lambda := g_*$ is an isometry of the intersection forms. The fact that Λ is a morphism in the sense of Boyer follows by replacing the third column in the above diagram by the relative homology groups and noting that this new diagram commutes by naturality. Note that the existence of an isometry of the intersection forms implies that all of the manifolds Z_n are spin since they have even intersection forms (Z having the stated intersection form can be verified by looking at the Kirby diagram in Figure 3). Boyer's second obstruction vanishes if we can show that the unique spin structure on Z_n and the unique spin structure on Z restrict to the same spin structure on Y. To show this, we will now be more explicit about the gluing maps used in the knot surgery.

We have an identification $\partial(E(K) \times \mathbb{S}^1) \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ where a longitude to K is identified with the third $\mathbb{S}^1$-factor, a meridian to K is identified with the first $\mathbb{S}^1$-factor, and the remaining $\mathbb{S}^1$-factor is identified with the second $\mathbb{S}^1$-factor. Similarly, we have an identification $\partial(Z \setminus \nu T) \cong T \times \mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ in the obvious way. This gives an identification $\partial(E(K) \times \mathbb{S}^1) \cong \partial(Z \setminus \nu T)$ and we use this to perform the knot surgery. Consider the unique spin structure σ on Z restricted to $\partial(Z \setminus \nu T) \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Note that σ extends over $\nu T \cong T \times \mathbb{D}^2$, and hence there are four possibilities for $\sigma|_{\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1}$. Conversely, there are four choices of spin structures on $E(K) \times \mathbb{S}^1$, and, by restricting to the boundary, these give rise to four distinct spin structures on $\partial(E(K) \times \mathbb{S}^1) \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Using the degree one map $E(K) \times \mathbb{S}^1 \to T \times \mathbb{D}^2$, we see that these are precisely the four spin structures which extend over $T \times \mathbb{D}^2$, and so regardless of how σ restricts to $\partial(Z \setminus \nu T)$, we can pick a spin structure on $E(K) \times \mathbb{S}^1$ such that $\sigma|_{Z\setminus\nu T}$ extends to a spin structure on Z_n. By construction, these two spin structures clearly match on Y, and hence we have a homeomorphism $Z \to Z_n$ which restricts to the identity map on Y. □

Theorem 4.4. *The 4-manifolds $\{Z_n\}_{n\in\mathbb{N}}$ are all homeomorphic relative to Y, but pairwise not diffeomorphic relative to Y. They are all simply-connected with intersection form $(\mathbb{Z}^2 \oplus \mathbb{Z}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right] \oplus 0)$ and have infinite* $\operatorname{Tor}(Z_n, Y)$. *Moreover, all non-trivial elements of the Torelli group* $\operatorname{Tor}(Z_n, Y)$ *are non-smoothable.*

Proof. The first part of the first statement follows directly from Proposition 4.3. In Figure 3(b) we depict an embedding of Z into $X := K3\#2\overline{\mathbb{CP}}^2$, whose Kirby diagram has been taken from [GS99, Fig. 8.16] (see also [AKMR15]). Hence Z_n embeds into the closed manifold X_n obtained by performing knot surgery on $T \hookrightarrow X$ using $K(n)$. From [FS98] and the blow-up formula [FS95], it follows that the manifolds X_n are pairwise non-diffeomorphic. Indeed, the Seiberg-Witten invariant of X_n, seen as an element of the group ring $\mathbb{Z}[H^2(X_n)]$, is equal to

$$SW(X_n) = (E_1 + E_1^{-1})(E_2 + E_2^{-1})\left(-(2n-1) + n(F^2 + F^{-2})\right), \tag{4.3}$$

where $E_i \in H^2(X_n)$ are the classes coming from the two blow-ups and $F := \mathrm{PD}[T]$ is the Poincaré dual to the torus T. Now, since X_n is obtained by capping Z_n with a fixed manifold $Q := X \setminus Z$ independent from n, the manifolds Z_n are pairwise non-diffeomorphic relative to their boundaries.

It remains to prove the last statement of the theorem. We want to apply Lemma 3.1, so we check that the hypotheses hold. We have that $H_1(Y)$ is isomorphic to $\mathbb{Z}^2$ generated by $v_1 := \mu_2 + \mu_3$ and $v_2 := \mu_5$, where the μ_i are the meridians to the components as shown in Figure 3(a). From (4.3) we see that $E_1 + E_2 \in H_2(Z_n)$ (E_1E_2 in group ring notation) is a Seiberg-Witten basic class for X_n, which restricts to $\partial \circ \mathrm{PD}(E_1 + E_2) = \mu_3 + \mu_5 = -2(v_1 + v_2)$ in Y. Moreover, the complement Q is obtained by adding only 2-handles and a single 4-handle to Y and hence $H_3(X_n, Z_n) \cong H_3(Q, Y) \cong H^1(Q) = 0$. Now the final statement of the theorem follows from Lemma 3.1. □

5. GENERALISED DEHN TWISTS

5.1. **Absolute non-smoothability and generalised Dehn twists.** We begin by reviewing absolute and relative smoothability from the point of view of spaces of maps. Given a smooth, compact, oriented manifold X with boundary (not necessarily of dimension four) we denote by $\mathrm{Diff}^+(X)$ the set of orientation-preserving self-diffeomorphisms of X topologised with the C^∞-topology, and by $\mathrm{Diff}^+(X, \partial X) \subset \mathrm{Diff}^+(X)$ the subspace of diffeomorphisms restricting to the identity over ∂X. Similarly we define $\mathrm{Homeo}^+(X)$ and $\mathrm{Homeo}^+(X, \partial X)$ using the compact-open topology. The inclusion map

$$(\mathrm{Diff}^+(X), \mathrm{Diff}(X, \partial X)) \to (\mathrm{Homeo}^+(X), \mathrm{Homeo}^+(X, \partial X))$$

is continuous. We will denote by

$$\begin{aligned}\Phi &: \pi_0 \mathrm{Diff}^+(X) \to \pi_0 \mathrm{Homeo}^+(X)\\ \Phi_\partial &: \pi_0 \mathrm{Diff}^+(X, \partial X) \to \pi_0 \mathrm{Homeo}^+(X, \partial X)\end{aligned}$$

the induced maps on the mapping class groups.

With these definitions in place, (relative) non-smoothability can be given an alternative definition by saying that $\varphi \in \pi_0 \mathrm{Homeo}^+(X, \partial X)$ is non-smoothable if $\varphi \notin \mathrm{im}\,\Phi$.

Definition 5.1. Let $i\colon \pi_0 \mathrm{Homeo}^+(X, \partial X) \to \pi_0 \mathrm{Homeo}^+(X)$ be the map induced by the inclusion. We say that $\varphi \in \pi_0 \mathrm{Homeo}^+(X, \partial X)$ is *absolutely* non-smoothable if $i(\varphi) \in \pi_0 \mathrm{Homeo}^+(X)$ does not belong to $\mathrm{im}\,\Phi$.

Explicitly, the difference between a relatively and an absolutely smoothable homeomorphism is that in the latter case the isotopy at each time does not need to fix the boundary pointwise. Surprisingly, for 4-dimensional manifolds the two notions of absolute and relative non-smoothability coincide. An important role in the proof is played by generalised Dehn twists [OP23, Sec. 1.2], which we now review.

Definition 5.2. Let X be a compact, smooth, oriented n-manifold with boundary. Given $[\gamma] \in \pi_1 \mathrm{Diff}^+(\partial X)$, we define the *generalised Dehn twist* with respect to $[\gamma]$ to be the smooth isotopy class of the diffeomorphism $\varphi_\gamma\colon X \to X$ defined on a collar of ∂X as $\varphi(y, t) = (\gamma(t)(y), t) \in (\partial X) \times I$ and defined outside of the collar as the identity map.

Another point of view is the following. The sequence of inclusion and restriction

$$\mathrm{Diff}^+(X, \partial X) \rightarrowtail \mathrm{Diff}^+(X) \to \mathrm{Diff}^+(\partial X)$$

and the equivalent sequence in the topological category are fibration sequences [Las76]. It can be shown that the connecting morphism $\pi_1 \mathrm{Diff}^+(\partial X) \to \pi_0 \mathrm{Diff}^+(X, \partial X)$ of the long exact sequence of homotopy groups is precisely the map that associates to a loop of diffeomorphisms $[\gamma]$ its generalised Dehn twist $[\varphi_\gamma]$ [OP23, Sec. 1.4].

Lemma 5.3. *Let X be a compact, smooth, oriented, 4-manifold with boundary. Then a mapping class $\varphi \in \pi_0 \mathrm{Homeo}^+(X, \partial X)$ is (relatively) non-smoothable if and only if it is absolutely non-smoothable.*

Proof. We will prove that relative non-smoothability implies absolute non-smoothability, the other implication is clear.

We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\pi_1 \mathrm{Diff}^+(\partial X) \arrow[r] \arrow[d, equal] & \pi_0 \mathrm{Diff}^+(X, \partial X) \arrow[r, "i"] \arrow[d, "\Phi_\partial"] & \pi_0 \mathrm{Diff}^+(X) \arrow[r, "\partial"] \arrow[d, "\Phi"] & \pi_0 \mathrm{Diff}^+(\partial X) \arrow[d, equal] \\
\pi_1 \mathrm{Diff}^+(\partial X) \arrow[r] & \pi_0 \mathrm{Homeo}^+(X, \partial X) \arrow[r, "i"] & \pi_0 \mathrm{Homeo}^+(X) \arrow[r, "\partial"] & \pi_0 \mathrm{Diff}^+(\partial X)
\end{tikzcd}$$

where i denotes the maps induced by the inclusion and we are implicitly using the well known homotopy equivalence $\mathrm{Homeo}(Y) \simeq \mathrm{Diff}(Y)$ for any 3-manifold Y [Cer59, Hat83]. Note that this is where we need the assumption that $\dim X = 4$.

Let $\phi \in \pi_0 \operatorname{Homeo}^+(X, \partial X)$ be non-smoothable. Suppose for a contradiction that there exists $\psi \in \pi_0 \operatorname{Diff}^+(X)$ such that $\Phi(\psi) = i(\phi)$. Then since $\partial(i(\varphi)) = \operatorname{Id}_{\partial X}$, the commutativity and exactness of the diagram implies that there exists $\psi' \in \pi_0 \operatorname{Diff}^+(X, \partial X)$ such that $i(\psi') = \psi$. Hence $\Phi_\partial(\psi')$ is equal to ϕ modulo composition with an element in the image of $\pi_1 \operatorname{Diff}^+(\partial X) \to \pi_0 \operatorname{Homeo}^+(X, \partial X)$. Thus $\Phi_\partial(\psi') = [\varphi_\gamma] \circ \phi$ for some generalised Dehn twist $[\varphi_\gamma]$, but then $\phi = [\varphi_\gamma]^{-1} \circ \Phi_\partial(\psi')$ presents ϕ as a composition of diffeomorphisms, contradicting the non-smoothability of ϕ. □

5.2. Realising smoothable elements of the Torelli group by generalised Dehn twists.

Since generalised Dehn twists are supported in a collar of the boundary, it is clear that these give rise to smooth elements in the Torelli group of the 4-manifold. One could ask whether generalised Dehn twists generate the whole Torelli group. The next proposition gives an answer under the assumption that the boundary is connected and prime; the general case is still unknown to the authors' best knowledge.

Proposition 5.4. *Let X be a smooth, compact, simply-connected, oriented 4-manifold with connected and* prime *boundary Y. Then the topological Torelli group* $\operatorname{Tor}(X, Y)$ *is realised by generalised Dehn twists if and only if one of the following holds:*

1. $b_1(Y) < 2$,
2. $b_2(Y) = 2$ *and Y is Seifert fibered with base orbifold* $\mathbb{T}^2$,
3. $Y = \mathbb{T}^3$,

where $\mathbb{T}^n$ denotes the n-torus.

Proof. We begin by showing that (1) or (3) implies that $\operatorname{Tor}(X, Y)$ is realised by generalised Dehn twists. First suppose that $b_1(Y) < 2$. Then $\Lambda^2 H_1(Y)^* = 0$ and hence $\operatorname{Tor}(X, Y)$ is trivial. The case $Y = \mathbb{T}^3$ can be handled by applying [OP23, Prop. 8.9] to the three generalised Dehn twists induced by the three $\mathbb{S}^1$-factors. More precisely, let $\alpha_1, \alpha_2, \alpha_3$ be the basis of $H_1(Y)$ induced by the $\mathbb{S}^1$-factors of $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ and let $\alpha_1^*, \alpha_2^*, \alpha_3^* \in H_1(Y)^*$ be the dual basis. Then the element of $\Lambda^2 H_1(Y)^*$ associated to a rotation of the i-th $\mathbb{S}^1$-factor is $\pm\alpha_k^* \wedge \alpha_j^*$, where $k, j \neq i$ [OP23, Prop.8.9], and hence the three rotations generate the whole Torelli group.

We now show that if neither (1) nor (3) hold, then either (2) holds or $\operatorname{Tor}(X, Y)$ is not realised by generalised Dehn twists. So assume that $Y \neq \mathbb{T}^3$ and $b_1(Y) \geq 2$. In this case Y is Haken [Wal68, 1.1.6] (therein called *sufficiently large*), and [Hat76] implies that $\pi_1 \operatorname{Diff}(Y) \cong Z(\pi_1(Y))$, hence in particular is abelian. Then it follows from [Wal67, Satz 4.1] that either the center $Z(\pi_1(Y))$ is trivial or Y is Seifert fibered over an orientable orbifold. In the former case, $b_1(Y) \geq 2$ implies that the Torelli group, being non-trivial, cannot be generated by generalised Dehn twists. In the latter case, $Z(\pi_1(Y)) \cong \mathbb{Z}$ generated by a principal orbit of the $\mathbb{S}^1$-action [Wal67], hence $\pi_1 \operatorname{Diff}(Y) \to \operatorname{Tor}(X, Y)$ cannot be surjective if $b_1(Y) > 2$, for in this case $\operatorname{Tor}(X, Y)$ has rank at least two.

We finish by showing that (2) implies that $\operatorname{Tor}(X, Y)$ is realised by generalised Dehn twists. When $b_1(Y) = 2$ and Y is Seifert fibered over an orientable orbifold, the quotient is necessarily $\mathbb{T}^2$ [BLPZ03]. Moreover the variation associated to the $\mathbb{S}^1$-action is computed in [OP23, Prop. 8.9] and in this case it generates the whole of $\Lambda^2 H_1(Y)^* \cong \mathbb{Z}$. □

In particular, if the boundary satisfies any of the three conditions of Proposition 5.4 then it is impossible to find a non-smoothable homeomorphism in the Torelli group.

Given the existence of non-smoothable elements of the Torelli group, we can say more. It is possible to find smoothable elements of the Torelli group which are not isotopic to any diffeomorphism supported on a collar of the boundary, let alone are realised by generalised Dehn twists.

To state the next theorem, recall that given two homeomorphisms of connected 4-manifolds $f\colon X_1 \to X_1$ and $g\colon X_2 \to X_2$, we can form the connect-sum homeomorphism $f\#g$ by first performing isotopies of f and g such that they restrict to the identity map on the discs used to perform the connect-sum. This fact follows from isotopy extension [EK71], uniqueness of normal bundles [FQ90, Chapter 9.3], and the fact that any orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ is isotopic to the identity [Cer68].

Theorem 5.5. *Let X be a smooth, simply-connected, oriented, compact 4-manifold with boundary such that there exists a non-smoothable self-homeomorphism $\varphi \in \operatorname{Tor}(X, \partial X)$. Then there exists an integer $m \geq 1$ such that*

$$\varphi \# \operatorname{Id}\colon X\#m(\mathbb{S}^2 \times \mathbb{S}^2) \to X\#m(\mathbb{S}^2 \times \mathbb{S}^2)$$

is a smoothable homeomorphism not isotopic to any smooth map supported on a collar of the boundary.

The proof relies on the following result.

Lemma 5.6. *Let X be a smooth, simply-connected, compact, oriented 4-manifold with boundary and $\varphi\colon X \to X$ a self-homeomorphism. Then there exists an integer $m \geq 1$ such that*

$$\psi := \varphi \# \operatorname{Id}\colon X\#m(\mathbb{S}^2 \times \mathbb{S}^2) \to X\#m(\mathbb{S}^2 \times \mathbb{S}^2)$$

is isotopic to a diffeomorphism relative to the boundary.

This result is proved in [FQ90, Sec. 8.6]. For more details on the proof, see [Gal].

Proof of Theorem 5.5. Let $\varphi \in \operatorname{Tor}(X, \partial X)$ be one of the non-smoothable mapping classes. By Lemma 5.6 there exists an integer $m \geq 1$ such that

$$\psi := \varphi \# \operatorname{Id}\colon X\#m(\mathbb{S}^2 \times \mathbb{S}^2) \to X\#m(\mathbb{S}^2 \times \mathbb{S}^2)$$

is isotopic to a diffeomorphism. Since ψ was defined by extending φ via the identity onto the $\mathbb{S}^2 \times \mathbb{S}^2$ summands, we also have that $\psi \in \operatorname{Tor}(X\#m(\mathbb{S}^2 \times \mathbb{S}^2))$. Now assume for a contradiction that ψ is supported on a collar of $\partial X\#m(\mathbb{S}^2 \times \mathbb{S}^2) \cong \partial X$. Then we can remove the $\mathbb{S}^2 \times \mathbb{S}^2$ summands and obtain a diffeomorphism $\psi'\colon X \to X$. However, since we have an identification

$$\operatorname{Tor}(X, \partial X) \cong \operatorname{Tor}(X\#m(\mathbb{S}^2 \times \mathbb{S}^2))$$

we can see that $\Delta_{\psi'} = \Delta_{\varphi}$. Hence, by Theorem 2.7 we see that ψ' and φ must be isotopic relative to the boundary. This contradicts the assumption that φ was not isotopic to a diffeomorphism, and so we conclude that ψ is not isotopic to any smooth map supported on a collar of the boundary. □

As an immediate corollary, we have that there exist examples of 4-manifolds with boundary where all elements of the Torelli group are smoothable, but all non-trivial elements are not supported on a collar of the boundary. This was Theorem 1.2 from the introduction.

Corollary 5.7. *There exists an infinite family of smooth, compact, oriented, simply-connected 4-manifolds with connected boundary $(W_n, \partial W_n)$ and $\operatorname{Tor}(W_n, \partial W_n)$ infinite order such that all mapping classes in $\operatorname{Tor}(W_n, \partial W_n)$ are smoothable, but only the identity map is supported* *on a collar of the boundary and, in particular, only the identity map is realised by a generalised Dehn twist.*

Proof. Apply Theorem 5.5 to the family X_n from Theorem 4.1. □

REFERENCES

[Akb16] Selman Akbulut. *4-manifolds*, volume 25. Oxford University Press, 2016.

[AKMR15] Dave Auckly, Hee Jung Kim, Paul Melvin, and Daniel Ruberman. Stable isotopy in four dimensions. *J. London Math. Soc.*, 91(2):439–463, 2015.

[Bar21] David Baraglia. Constraints on families of smooth 4-manifolds from Bauer-Furuta invariants. *Algebr. Geom. Topol.*, 21(1):317–349, 2021.

[BF04] Stefan Bauer and Mikio Furuta. A stable cohomotopy refinement of Seiberg-Witten invariants: I. *Invent. Math.*, 155(1):1–19, 2004.

[BLPZ03] J. Bryden, T. Lawson, B. Pigott, and P. Zvengrowski. The integral homology of orientable Seifert manifolds. *Topol. Appl.*, 127(1):259–275, 2003. Proceedings of the Pacific Institute for the Mathematical Sciences workshop "Invariants of Three-Manifolds".

[Boy86] Steven Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Am. Math. Soc.*, 298:331–357, 1986.

[Cer59] Jean Cerf. Groupes d'automorphismes et groupes de difféomorphismes des variétés compactes de dimension 3. *Bull. Soc. Math. Fr.*, 87:319–329, 1959.

[Cer68] Jean Cerf. *Sur les difféomorphismes de la sphère de dimension trois* $\Gamma_4 = 0$, volume 53 of *Lect. Notes Math.* Springer, Cham, 1968.

[Don90] S. K. Donaldson. Polynomial invariants for smooth four-manifolds. *Topology*, 29(3):257–315, 1990.

[EK71] R. D. Edwards and R. C. Kirby. Deformations of spaces of imbeddings. *Ann. Math. (2)*, 93:63–88, 1971.

[Eli04] Yakov Eliashberg. A few remarks about symplectic filling. *Geom. Topol.*, 8(1):277 – 293, 2004.

[EMM22] John B. Etnyre, Hyunki Min, and Anubhav Mukherjee. On 3-manifolds that are boundaries of exotic 4-manifolds. *Trans. Am. Math. Soc.*, 375(6):4307–4332, June 2022.

[Etn04] John B. Etnyre. On symplectic fillings. *Algebr. Geom. Topol.*, 4:73–80, 2004.

[FM88] Robert Friedman and John W. Morgan. On the diffeomorphism types of certain algebraic surfaces. I. *J. Differ. Geom.*, 27(2):297–369, 1988.

[FQ90] Michael H. Freedman and Frank S. Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Math. Ser.* Princeton, NJ: Princeton University Press, 1990.

[Fre82] Michael H. Freedman. The topology of four-dimensional manifolds. *J. Differ. Geom.*, 17:357–453, 1982.

[FS95] Ronald Fintushel and Ronald J. Stern. Immersed spheres in 4-manifolds and the immersed Thom conjecture. *Turk. J. Math.*, 19(2):145–157, 1995.

[FS98] Ronald Fintushel and Ronald J. Stern. Knots, links, and 4-manifolds. *Invent. Math.*, 134(2):363–400, 1998.

[Gal] Daniel A.P. Galvin. The Casson-Sullivan invariant for self-homeomorphisms of 4-manifolds. In preparation.

[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.

[Gom91] Robert E. Gompf. Nuclei of elliptic surfaces. *Topology*, 30(3):479–511, 1991.

[Gom98] Robert E. Gompf. Handlebody construction of Stein surfaces. *Ann. Math.*, 148(2):619–693, 1998.

[GS99] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society, 1999.

[Hat76] Allen E. Hatcher. Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds. *Topology*, 15(4):343–347, 1976.

[Hat83] Allen E. Hatcher. A proof of the Smale conjecture, $\mathrm{Diff}(S^3) \simeq O(4)$. *Ann. Math.*, 117(3):553–607, 1983.

[KLS18] Tirasan Khandhawit, Jianfeng Lin, and Hirofumi Sasahira. Unfolded Seiberg–Witten Floer spectra, I: Definition and invariance. *Geom. Topol.*, 22(4):2027 – 2114, 2018.

[KM07] P.B. Kronheimer and T.S. Mrowka. *Monopoles and Three-Manifolds.* New Mathematical Monographs. Cambridge University Press, 2007.

[KT22] Hokuto Konno and Masaki Taniguchi. The groups of diffeomorphisms and homeomorphisms of 4-manifolds with boundary. *Adv. Math.*, 409 A:58, 2022. Id/No 108627.

[Las76] R. Lashof. Embedding spaces. *Ill. J. Math.*, 20:144–154, 1976.

[LM97] P. Lisca and G. Matić. Tight contact structures and Seiberg–Witten invariants. *Invent. Math.*, 129(3):509–525, 1997.

[Man03] Ciprian Manolescu. Seiberg-Witten-Floer stable homotopy type of three-manifolds with $b_1 = 0$. *Geom. Topol.*, 7(2):889–932, 2003.

[MMP20] Ciprian Manolescu, Marco Marengon, and Lisa Piccirillo. Relative genus bounds in indefinite four-manifolds. 2020.

[MS97] John W. Morgan and Zoltán Szabó. Homotopy $K3$ surfaces and mod 2 Seiberg-Witten invariants. *Math. Res. Lett.*, 4(1):17–21, 1997.

[OP23] Patrick Orson and Mark Powell. Mapping class groups of simply connected 4-manifolds with boundary. Preprint available at https://arxiv.org/pdf/2207.05986.pdf, 2023.

[OS04] Burak Özbağcı and András I. Stipsicz. *Surgery on contact 3-manifolds and Stein surfaces*, volume 13 of *Bolyai Society Mathematical Studies*. Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004.

[OS06] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math.*, 202(2):326–400, 2006.

[Per86] Bernard Perron. Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique. *Topology*, 25(4):381–397, 1986.

[Qui86] Frank S. Quinn. Isotopy of 4-manifolds. *J. Differ. Geom.*, 24:343–372, 1986.

[RS23] Daniel Ruberman and Sašo Strle. Wall's stable realization for diffeomorphisms of definite 4-manifolds. Preprint available at https://arxiv.org/pdf/2210.16260.pdf, 2023.

[Sae06] Osamu Saeki. Stable mapping class groups of 4-manifolds with boundary. *Trans. Am. Math. Soc.*, 358(5):2091–2104, 2006.

[SS21] Hirofumi Sasahira and Matthew Stoffregen. Seiberg-Witten Floer spectra for $b_1 > 0$. Preprint available at https://arxiv.org/pdf/2103.16536.pdf, 2021.

[Tau94] Clifford H. Taubes. The Seiberg-Witten invariants and symplectic forms. *Math. Res. Lett.*, 1:809–822, 1994.

[Wal64] C. T. C. Wall. Diffeomorphisms of 4-manifolds. *J. Lond. Math. Soc.*, 39:131–140, 1964.

[Wal67] Friedhelm Waldhausen. Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten. *Topology*, 6(4):505–517, 1967.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. Math.*, 87(1):56–88, 1968.

[Wit94] Edward Witten. Monopoles and four manifolds. *Math. Res. Lett.*, 1:769–796, 1994.

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