NON-SMOOTHABLE SURFACES IN THE 4-SPHERE

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Abstract. We construct examples of non-smoothable surfaces in the 4-sphere, thereby answering Question 4.32 on the K3 problem list. These surfaces are non-orientable and have knot group of order 2, thus simultaneously answering Question 4.29(a) on the K3 problem list.

1. Introduction

Question 4.32 on the K3 problem list asks whether there exist non-smoothable surfaces in S^4, i.e. locally flat surfaces that are not isotopic to any smoothly embedded surface [BKR26]. There are such surfaces in 4-manifolds other than S^4 [Kug84, Rud84, Luo88, LW97, Tor25]; see also [Las71] in high dimensions. We construct infinitely many non-smoothable surfaces in S^4, each of which is a $\mathbb{Z}_2$-*surface*, meaning a non-orientable surface $F \subset S^4$ with $\pi_1(S^4 \setminus F) \cong \mathbb{Z}_2$.

Theorem 1.1. *There are infinitely many non-smoothable knotted $\mathbb{Z}_2$-surfaces in S^4, pairwise distinct up to isotopy.*

Here, a non-orientable surface in S^4 is *unknotted* if it is isotopic to a connected sum of unknotted projective planes. In turn, a projective plane in S^4 is *unknotted* if it is isotopic to the branch locus of one of the branched covers $\mathbb{C}P^2 \to S^4$ or $\overline{\mathbb{C}P}^2 \to S^4$ induced by complex conjugation.

Theorem 1.1 also answers in the negative the question of whether every $\mathbb{Z}_2$-surface in S^4 is topologically unknotted; this is Question 4.29(a) on the K3-problem list [BKR26]. The question also appeared in e.g. [Kaw96, p. 181], [Kam02, p. 55], [BCD+21, p. 7] and [COP23, Question 1.1]. For context, it is known that a $\mathbb{Z}_2$-surface $F \subset S^4$ with non-orientable genus h and normal Euler number e is unknotted when $|e| \neq 2h$ [COP23, Theorem A]. The same holds for $h \leq 5$ and $|e| = 2h$ (with the case $h = 1$ originally due to Lawson [Law84] and the cases $h = 4, 5$ requiring [Pen24]).

1.1. **Outline of the proof.** There are natural candidates for non-smoothable $\mathbb{Z}_2$-surfaces, namely those whose double branched cover have a definite but non-diagonalisable form. The challenge is to construct a surface realising such a form. Section 3 associates to every odd non-singular symmetric bilinear form $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ a non-degenerate form b^{nd}_{ext} that is the algebraic incarnation of the intersection form on the universal cover of the surface exterior; we call this form the *exterior non-degenerate form.* To this form Section 4 associates a $\mathbb{Z}[\mathbb{Z}_2]$-valued hermitian form λ_b we call the *exterior equivariant form* via a pullback construction due to Hambleton-Riehm [HR78]. We observe in Proposition 2.1, using the work of Freedman [Fre82, PRT25], that this latter form is realisable as the equivariant intersection form of a surface exterior if and only if it is stably realisable, that is realisable after taking the direct sum with some number of hyperbolic forms. Section 5 (and specifically Proposition 5.3) proves that λ_b is stably realisable by using the properties of the pullback. In short, we apply the pullback construction alongside the fact that an odd non-singular non-diagonalisable definite form b is stably isometric to a diagonalisable one. The outcome is a $\mathbb{Z}_2$-surface $F \subset S^4$ for which the (non-degenerate part of the) intersection form on the universal cover of the surface exterior is b^{nd}_{ext}. It then remains to show that the branched double cover $\Sigma_2(F)$ has intersection form $Q_{\Sigma_2(F)}$ isometric to b, and this is the only step where we encounter a potential obstruction. This obstruction is overcome with a mild restriction by using recent work of Chenevier on Kneser neighbours of the standard lattice $\mathbb{Z}^h$ [Kne57, Che25].

In summary, we will prove the following theorem.

Theorem 1.2. *If $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ is an odd definite non-singular symmetric bilinear form of rank $h \not\equiv 4 \mod 8$, then there exists a non-orientable $\mathbb{Z}_2$-surface $F \subset S^4$ such that $Q_{\Sigma_2(F)} = b$.*

We explain how Theorem 1.2 combines with work of Donaldson [Don83] to yield Theorem 1.1.

Proof of Theorem 1.1. Let b be a form as in Theorem 1.2 but not diagonalisable. By Theorem 1.2 there is a $\mathbb{Z}_2$-surface $F \subset S^4$ such that $Q_{\Sigma_2(F)} = b$. By Donaldson [Don83] b is not realised by a smooth 4-manifold and so $\Sigma_2(F)$ is non-smoothable. We deduce that $F \subset S^4$ is non-smoothable. To obtain infinitely many surfaces, observe that there are infinitely many such forms (for example $E_8 \oplus (1)^{\oplus n}$ with $n \geq 1$) and all of the non-smoothable surfaces constructed this way are distinguished pairwise via their branched double covers. □

Remark 1.3. The branched double covers $\Sigma_2(F)$ for the surfaces in the above theorems have trivial Kirby-Siebenmann invariant; this follows from the fact that if a closed oriented 4-manifold X supports an orientation preserving locally linear involution, then $\mathrm{ks}(X) = 0$ [KV86]; see also [Edm89, Remark on page 120]. Accordingly, in the simplest case of our theorem (for $b = E_8 \oplus (1)$) we have that $\Sigma_2(F) \cong E_8 \# {*}\mathbb{C}P^2$, where E_8 denotes Freedman's E_8 manifold and ${*}\mathbb{C}P^2$ denotes the Chern manifold, which is homotopy equivalent but not homeomorphic to $\mathbb{C}P^2$. This surface $F \subset S^4$ has non-orientable genus 9 and normal Euler number -18.

Remark 1.4. We make a note on why we are unable to decide on the existence of a $\mathbb{Z}_2$-surface in the case that the target form b has rank $h \equiv 4 \mod 8$. The first step of our proof forgets the actual form b and instead considers the form b^{nd}_{ext} which is a certain index 2 subform of b. There are examples of non-isomorphic forms who have isomorphic index 2 subforms. Due to this, we are unable a priori to deduce that our built surface F has $Q_{\Sigma_2(F)} \cong b$. In all cases where $h \not\equiv 4 \mod 8$, we appeal to work of Chenevier [Che25] to bypass this problem.

In light of the above remark, we record the first (in terms of rank) incarnation of our failure in the case where $h \equiv 4 \mod 8$. Let Γ_{8k+4} denote the rank $8k+4$ odd definite non-singular symmetric bilinear form defined in e.g. [Ser78, Chapter V, §1.4]. The form Γ_{12} satisfies $(\Gamma_{12})^{nd}_{ext} \cong ((1)^{\oplus 12})^{nd}_{ext}$ (see [Che25, Section 1.4]), and so our proof in this case cannot conclude that the surface we produce is knotted.

Question 1.5. Does the form Γ_{12} occur as the intersection form of the branched double cover of a $\mathbb{Z}_2$-surface in S^4?

There are three isometry classes of positive definite non-singular symmetric bilinear forms of rank 12, namely $(1)^{\oplus 12}, \Gamma_{12}$ and $E_8 \oplus (1)^{\oplus 4}$; see [MH73, Chapter II, Lemma 6.2 and Remark 1] or [Che25, Table 1]. Despite not knowing whether Γ_{12} can be realised, we are able to find a $\mathbb{Z}_2$-surface with $Q_{\Sigma_2(F)} = E_8 \oplus (1)^{\oplus 4}$. In fact, applying this reasoning to $E_8 \oplus (1)^{\oplus k}$ for every $k \geq 1$ produces a non-smoothable knotted $\mathbb{Z}_2$-surface $F \subset S^4$ of non-orientable genus $8+k$. For $k = 8\ell+4$ with $\ell > 1$, since the number of positive definite non-singular symmetric bilinear forms of a given rank grows rapidly, we however cannot ensure that $Q_{\Sigma_2(F)} = E_8 \oplus (1)^{\oplus k}$, only that $Q_{\Sigma_2(F)}$ is not diagonalisable. Nevertheless, combining this with the unknotting theorem from [COP23, Theorem A] and Massey's characterisation of integers that arise as the normal Euler number of a non-orientable surface in S^4 [Mas69], we obtain the following result.

Theorem 1.6. *For every $h \geq 9$ and every $e \in \{-2h, -2h+4, \ldots, 2h-4, 2h\}$, there exists a knotted $\mathbb{Z}_2$-surface of non-orientable genus h and Euler number e if and only if $|e| = 2h$.*

Clearly Theorem 1.2 implies the "if direction" of Theorem 1.6 unless $h \equiv 4 \mod 8$, and so this is the only case where Theorem 1.6 is of distinct interest. Since $\mathbb{Z}_2$-surfaces of non-orientable genus $h \leq 5$ are unknotted [COP23, Pen24], Theorem 1.6 addresses the question of whether $\mathbb{Z}_2$-surfaces of non-orientable genus h are unknotted for every $h \neq 6, 7, 8$.

1.2. **Further context and comparisons.** A driving question in the study of knotted surfaces asks whether an embedded surface in S^4 with cyclic knot group is necessarily unknotted. Here, note the knot group is $\mathbb{Z}$ or $\mathbb{Z}_2$ according to whether or not the surface is orientable. In the topological category, the answer is positive for orientable surfaces of genus $g \neq 1, 2$ [FQ90, CP23] (and remains open for $g = 1, 2$) as well as for non-orientable surfaces when the Euler number is non-extremal or when the non-orientable genus is at most 5 [COP23, Pen24]; see also [Law84, FKV88, Kre90] for early work on the topic. In fact, in the topological category, the aforementioned question was often phrased as the conjecture that locally flat surfaces in S^4 with cyclic knot group are unknotted, see e.g. [BCD+21, BKR26]. The question remains open in the case of non-orientable genus $h = 6, 7, 8$. In the smooth category, the question remains open in the orientable case, but admits a negative answer in the non-orientable case [FKV88, Fin02, Miy23, MOJ+23].

Organisation. Section 2 recalls some facts about intersection forms and notes that the realisation problem is a stable question. Section 3 introduces the exterior non-degenerate form. Section 4 recalls Hambleton-Riehm's pullback construction of hermitian forms. Section 5 proves Theorems 1.2 and 1.6. Appendix A contains a brief discussion of lattices and Kneser neighbours. Appendix B proves a root-counting argument that is used during the proof of Theorem 1.6.

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Conventions. We work in the topological category with locally flat embeddings. Manifolds are assumed to be compact, connected and oriented unless otherwise specified. Modules are assumed to be finitely generated. To simplify notation the proofs are written with all definite forms considered to be positive. The interested reader can readily extend all herein arguments to negative definite forms by substituting the non-orientable genus h unknot with Euler number $-2h$ for the unknot with Euler number $+2h$ and making the corresponding subsequent modifications.

2. Realisation and stable realisation of non-degenerate forms

Given a finite group π, this section is concerned with the problem of realising a $\mathbb{Z}[\pi]$-valued hermitian form as the equivariant intersection form of a 4-manifold with a prescribed boundary and fundamental group π. After introducing some terminology, we show that this is essentially a stable problem (Proposition 2.1).

Fix a closed 3-manifold Y and an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \pi$ to a finite group π. We say that a 4-manifold X with $\pi_1(X) \cong \pi$ is a *filling of* (Y, φ) if there is a homeomorphism $\partial X \cong Y$ such that the composition $\pi_1(Y) \cong \pi_1(\partial X) \to \pi_1(X) \cong \pi$ agrees with φ. In what follows, given a filling X of (Y, φ) with $\pi_1(X) \cong \pi$, we write $\widetilde{X}$ for its universal cover, $Y^\varphi = \partial\widetilde{X}$ for its boundary, $Q_{\widetilde{X}}$ for the $\mathbb{Z}$-intersection form on $H_2(\widetilde{X})$, and $Q^{nd}_{\widetilde{X}}$ for the induced non-degenerate form on $\operatorname{coker}(i_*\colon H_2(Y^\varphi) \to H_2(\widetilde{X}))$. Here, we used that

$$\left\{x \in H_2(\widetilde{X}) \mid Q_{\widetilde{X}}(x, y) = 0 \text{ for all } y \in H_2(\widetilde{X})\right\} = \operatorname{im}(i_*\colon H_2(Y^\varphi) \to H_2(\widetilde{X})).$$

The left hand side of this equation is referred to as the *radical* of $Q_{\widetilde{X}}$ and is abbreviated as rad. Since $\pi_1(\partial X) \to \pi_1(X)$ is surjective (so that $H_3(\widetilde{X}) \cong H^1(\widetilde{X}, \partial\widetilde{X}) = 0$), it follows that i_* is injective, whence rad $\cong H_2(Y^\varphi)$.

We note that $\operatorname{im}(i_*)$ is also the radical of the $\mathbb{Z}[\pi]$-equivariant intersection form

$$\lambda_X\colon H_2(\widetilde{X}) \times H_2(\widetilde{X}) \to \mathbb{Z}[\pi].$$

A *stabilisation* of a $\mathbb{Z}[\pi]$-hermitian form (H, λ) refers to the hermitian form $(H, \lambda) \oplus \left(\mathbb{Z}[\pi]^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$. We say that (H, λ) and (H', λ') are *stably isometric* if they become isometric after each pair is stabilised n times for some $n \geq 0$. A $\mathbb{Z}[\pi]$-hermitian form (H, λ) is *realisable* if there is a filling X of (Y, φ) with $\pi_1(X) \cong \pi$ and $(H_2(\widetilde{X}), \lambda_X) \cong (H, \lambda)$. A form (H, λ) is *stably realisable* if (H, λ) becomes realisable after some number of stabilisations.

The following proposition is similar to [HK88, Lemma 4.1].

Proposition 2.1. *Fix a closed* 3*-manifold* Y *and an epimorphism* $\varphi\colon \pi_1(Y) \twoheadrightarrow \pi$ *to a finite group* π*. A* $\mathbb{Z}[\pi]$*-hermitian form* (H, λ) *is realisable by a filling of* (Y, φ) *if and only if it is stably realisable by a filling of* (Y, φ)*.*

Proof. A realisable form is certainly stably realisable, so we focus on the converse. Assume there is a filling X of (Y, φ) with $\pi_1(X) \cong \pi$, and an isometry

$$(H, \lambda) \oplus \left(\mathbb{Z}[\pi]^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)^{\oplus n} \cong (H_2(\widetilde{X}), \lambda_X).$$

Since finite groups are good (see e.g. [BKK+21, Theorem 19.2]) and $H_2(\widetilde{X}) \cong \pi_2(X)$, the sphere embedding theorem [FQ90] now implies that X splits off n copies of $S^2 \times S^2$; see e.g. [PRT25, Theorem 2.3]; the condition involving self-intersections can be omitted because X is orientable. Surgering these connected summands yields a filling of (Y, φ) that realises (H, λ). □

3. The exterior non-degenerate form

This section introduces an algebraic construction that captures how, given a $\mathbb{Z}_2$-surface $F \subset S^4$ with exterior X_F, the non-degenerate $\mathbb{Z}$-intersection form on the universal cover of $\widetilde{X}_F$ can be recovered from the intersection form of the double branched cover.

Definition 3.1. The *exterior non-degenerate form* b^{nd}_{ext} associated to a non-singular symmetric bilinear form (H, b) over $\mathbb{Z}$ refers to the pair $(H^{nd}_{ext}, b^{nd}_{ext})$, where

- the $\mathbb{Z}[\mathbb{Z}_2]$-module H^{nd}_{ext} is defined, as an abelian group, as
$$H^{nd}_{ext} := \Big\{ x \in H \mid b(x,x) = 0 \mod 2 \Big\} \subset H$$
and is endowed with the $\mathbb{Z}[\mathbb{Z}_2]$-module structure induced by $Tx = -x$ for every $x \in H^{nd}_{ext}$.
- The non-degenerate symmetric bilinear form b^{nd}_{ext} is the restriction of b to H^{nd}_{ext}.

If a non-singular symmetric bilinear form b is odd, then the map $H \to \mathbb{Z}_2, x \mapsto b(x,x) \mod 2$ is surjective and leads to the short exact sequence
$$0 \to H^{nd}_{ext} \xrightarrow{\subseteq} H \to \mathbb{Z}_2 \to 0.$$
In particular, $(H^{nd}_{ext}, b^{nd}_{ext})$ is an index 2 subform of (H, b).

Example 3.2. We illustrate this definition with some examples.

- If $(H, b) = (H_2(\Sigma_2(F)), Q_{\Sigma_2(F)})$ is the $\mathbb{Z}$-intersection form of the branched cover of a $\mathbb{Z}_2$-surface $F \subset S^4$, then, by [COP23, Proposition 5.10], the associated exterior form is the non-degenerate form $Q^{nd}_{\widetilde{X}_F}$ on $H_2(\widetilde{X}_F)/\operatorname{rad}$. Here, recall that $\widetilde{X}_F$ denotes the universal cover of the exterior of F. We record the precice statement for later use. First, [COP23, Proposition 5.10] shows that the inclusion $\widetilde{X}_F \hookrightarrow \Sigma_2(F)$ induces a form-preserving injection
$$(H_2(\widetilde{X}_F)/\operatorname{rad}, Q^{nd}_{\widetilde{X}_F}) \hookrightarrow (H_2(\Sigma_2(F)), Q_{\Sigma_2(F)}).$$
Then [COP23, Proposition 5.10] shows that the image of this injection is $(H^{nd}_{ext}, b^{nd}_{ext})$.
- If b is a non-singular symmetric bilinear form and H is a hyperbolic form over $\mathbb{Z}$, then
$$(b \oplus H)^{nd}_{ext} \cong b^{nd}_{ext} \oplus H.$$
This is because hyperbolics are even and therefore all squares are zero mod 2.
- When $(H, b) = (\mathbb{Z}^h, (1)^{\oplus h})$, the calculation from [COP23, Proof of Proposition 5.11] gives
$$(H^{nd}_{ext}, b^{nd}_{ext}) = \begin{pmatrix} 4 & 2 & \cdots & & & 2 \\ 2 & 2 & 1 & \cdots & & 1 \\ \vdots & 1 & 2 & 1 & \cdots & \vdots \\ & \vdots & 1 & \ddots & & \\ & & \vdots & & \ddots & 1 \\ 2 & 1 & \cdots & & 1 & 2 \end{pmatrix}. \tag{1}$$
In particular, this describes $Q^{nd}_{\widetilde{X}_F}$ when $F = U \subset S^4$ is the unknotted surface of non-orientable genus h and extremal Euler number $e = -2h$.

The next proposition records how this construction interacts with stabilisations.

Proposition 3.3. *If two non-singular symmetric bilinear forms b and b' are stably isometric, then their exterior forms $(H^{nd}_{ext}, b^{nd}_{ext})$ and $((H'_{ext})^{nd}, (b'_{ext})^{nd})$ are stably isometric.*

Proof. Assume that b and b' are stably isometric, say $b \oplus H \cong b' \oplus H'$ for some hyperbolic forms H and H'. It follows from the second item of Example 3.2 that

$$b^{nd}_{ext} \oplus H \cong (b \oplus H)^{nd}_{ext} \cong (b' \oplus H')^{nd}_{ext} \cong (b')^{nd}_{ext} \oplus H'.$$

Here, note that since the exterior forms are defined using the module structure induced by $Tx = -x$ for every x, this $\mathbb{Z}$-isometry is automatically a $\mathbb{Z}[\mathbb{Z}_2]$-isomorphism on the underlying $\mathbb{Z}[\mathbb{Z}_2]$-modules. This concludes the proof of the proposition. □

4. Pullbacks of hermitian forms over $\mathbb{Z}[\mathbb{Z}_2]$.

This section is concerned with pullbacks. Section 4.1 recalls a pullback construction due to Hambleton-Riehm [HR78] in the setting of hermitian forms over $\Lambda := \mathbb{Z}[\mathbb{Z}_2]$. Section 4.2 applies this construction to equivariant intersection forms of $\mathbb{Z}_2$-surface exteriors.

4.1. Pullbacks.

We begin with some notation.

Notation 4.1. Given a Λ-module M, consider the abelian groups $M_+ := \{x \in M \mid Tx = x\}$ and $M_- := \{x \in M \mid Tx = -x\}$. When M has no 2-torsion, it fits into the pullback square

$$\begin{array}{ccc} M & \longrightarrow & M/M_+ \\ \downarrow & & \downarrow \\ M/M_- & \longrightarrow & M/(M_+, M_-) := M_2. \end{array}$$

When $M = \Lambda$, we write $\mathbb{Z}_\pm := \Lambda/\Lambda_\mp = \Lambda/(1 \mp T)$ for the abelian group $\mathbb{Z}$ endowed with the $\mathbb{Z}_2$-action given by $Tx = \pm x$ for every $x \in \mathbb{Z}$. The previous pullback square then reduces to

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbb{Z}_- \\ \downarrow & & \downarrow \\ \mathbb{Z}_+ & \longrightarrow & \mathbb{Z}_2. \end{array}$$

We also write $\Lambda^\mathbb{Q} := \mathbb{Q}[\mathbb{Z}_2]$ so that for $M^\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$, the quotient maps induce an isomorphism

$$M^\mathbb{Q} \xrightarrow{\cong} M^\mathbb{Q}/M^\mathbb{Q}_- \oplus M^\mathbb{Q}/M^\mathbb{Q}_+.$$

In particular, taking $M = \Lambda$, we record the isomorphism

$$\Lambda^\mathbb{Q} \xrightarrow{\cong} \frac{\Lambda^\mathbb{Q}}{(1-T)} \oplus \frac{\Lambda^\mathbb{Q}}{(1+T)} =: \mathbb{Q}_+ \oplus \mathbb{Q}_-.$$

The inverse of this map is given by $([a], [b]) \mapsto ((1 + T)a + (1 - T)b)/2$.

Next, we move on from pullbacks of modules to pullbacks of hermitian forms. In what follows, we will assume that M is torsion-free as an abelian group so that a theorem of Reiner ensures that M decomposes as a direct sum of $\mathbb{Z}_+, \mathbb{Z}_-$ and Λ-summands [Rei57].

Construction 4.2. To a hermitian form $\lambda \colon M \times M \to \Lambda$ on a Λ-module M that is $\mathbb{Z}$-torsion-free as an abelian group, we associate symmetric bilinear forms

$$\begin{aligned} &\lambda_+ \colon M/M_- \times M/M_- \to \Lambda/\Lambda_- \cong \mathbb{Z}_+, \\ &\lambda_- \colon M/M_+ \times M/M_+ \to \Lambda/\Lambda_+ \cong \mathbb{Z}_-, \text{ and} \\ &\lambda_2 \colon M/(M_-, M_+) \times M/(M_-, M_+) \to \Lambda/(\Lambda_-, \Lambda_+) \cong \mathbb{Z}_2. \end{aligned}$$

Tensor the form (M, λ) by $\mathbb{Q}$ and denote the outcome by $(M^\mathbb{Q}, \lambda^\mathbb{Q})$. This form is $\Lambda^\mathbb{Q} := \mathbb{Q}[\mathbb{Z}_2]$-valued. The aforementioned quotient-induced isomorphism then leads to a decomposition

$$(M^\mathbb{Q}, \lambda^\mathbb{Q}) \cong (M^\mathbb{Q}/M^\mathbb{Q}_-, \lambda^\mathbb{Q}_+) \oplus (M^\mathbb{Q}/M^\mathbb{Q}_+, \lambda^\mathbb{Q}_-).$$

The situation is summarised by the following commutative diagram:

$$\begin{array}{ccc} M^{\mathbb{Q}} \times M^{\mathbb{Q}} & \xrightarrow{\quad \lambda^{\mathbb{Q}} \quad} & \Lambda^{\mathbb{Q}} \\ \downarrow \cong & & \downarrow \cong \\ (M^{\mathbb{Q}}/M^{\mathbb{Q}}_{-} \oplus M^{\mathbb{Q}}/M^{\mathbb{Q}}_{+}) \times (M^{\mathbb{Q}}/M^{\mathbb{Q}}_{-} \oplus M^{\mathbb{Q}}/M^{\mathbb{Q}}_{+}) & \xrightarrow{(\lambda^{\mathbb{Q}}_{+} \ \lambda^{\mathbb{Q}}_{-})} & \mathbb{Q}_{+} \oplus \mathbb{Q}_{-}. \end{array}$$

Write $\mathrm{proj}_{\pm}\colon M \to M/M_{\pm}$ for the canonical projections and set $x_{\pm} := \mathrm{proj}_{\mp}(x)$. Since M is $\mathbb{Z}$-torsion-free, restricting the pair of forms $(\lambda^{\mathbb{Q}}_{+}, \lambda^{\mathbb{Q}}_{-})$ to $M/M_{-} \oplus M/M_{+}$ then leads to a pair $(\lambda_{+}, \lambda_{-})$ of $\mathbb{Z}$-valued forms that are related to λ as follows:

$$\lambda(x,y) = (\lambda_{+}(x_{+}, y_{+}), \lambda_{-}(x_{-}, y_{-})). \tag{2}$$

These forms descend to $M_2 := M/(M_{-}, M_{+})$ and induce a symmetric bilinear form on M_2.

$$\lambda_2(x,y) := [\lambda_{+}([x_{+}],[y_{+}])] = [\lambda_{-}([x_{-}],[y_{-}])] \in \Lambda/(\Lambda_{-}, \Lambda_{+}) \cong \mathbb{Z}_2. \tag{3}$$

We briefly comment on the notation used in (2).

Remark 4.3. In (2), we followed the notation of Hambleton-Riehm but it it is worth spelling out the meaning of this equation for later use. Indeed, the left hand side belongs to Λ whereas the right hand side belongs to

$$\Lambda_{pull} := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \frac{\Lambda}{(1-T)} \oplus \frac{\Lambda}{(1+T)} \mid \mathrm{proj}_{+}(a) = \mathrm{proj}_{-}(b) \right\}.$$

The projections $\Lambda \to \Lambda/(1 \pm T)$ are seen to induce an isomorphism $\Lambda \xrightarrow{\cong} \Lambda_{pull}$ whose inverse is given by $(a,b) \mapsto ((1+T)a + (1-T)b)/2$; here the numerator of this expression is even because it can be written as $(a+b)+T(a-b)$, where $a \pm b$ is even thanks to the condition $\mathrm{proj}_{+}(a) = \mathrm{proj}_{-}(b)$. In particular, we record for later use that (2) can be rewritten as

$$\lambda(x,y) = \frac{(1+T)\lambda_{+}(x_{+}, y_{+}) + (1-T)\lambda_{-}(x_{-}, y_{-})}{2}. \tag{4}$$

The following result is due to Hambleton and Riehm [HR78, Theorems 3 and 6].

Theorem 4.4. *Fix a $\mathbb{Z}$-torsion-free module Λ-module M with associated pullback square*

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{proj}_{+}} & M/M_{+} \\ {\scriptstyle \mathrm{proj}_{-}} \downarrow & & \downarrow \\ M/M_{-} & \longrightarrow & M/(M_{+}, M_{-}) := M_2. \end{array}$$

The following assertions hold:

- *Given a hermitian form (M, λ), there are unique symmetric bilinear forms $(M/M_{-}, \lambda_{+})$ and $(M/M_{+}, \lambda_{-})$ satisfying (2) and a unique symmetric bilinear form (M_2, λ_2) satisfying (3).*
- *Given symmetric bilinear forms $(M/M_{-}, \lambda_{+}), (M/M_{+}, \lambda_{-}), (M_2, \lambda_2)$ satisfying (3), there exists a unique hermitian form λ on M satisfying (2).*

Since the form (M_2, λ_2) is determined by $(M/M_{+}, \lambda_{-}), (M/M_{-}, \lambda_{+})$ and (3), we will frequently omit it from the notation.

Example 4.5. Let $H(\Gamma)$ denote the standard hyperbolic form over Γ where $\Gamma = \mathbb{Z}[\mathbb{Z}_2], \mathbb{Z}_{-}, \mathbb{Z}_{+}$ or $\mathbb{Z}_2$. We prove that the pullback of the hyperbolic forms $H(\mathbb{Z}_{+})$ and $H(\mathbb{Z}_{-})$ is the hyperbolic form $H(\mathbb{Z}[\mathbb{Z}_2])$. By Theorem 4.4, it suffices to prove that for $\lambda := H(\mathbb{Z}[\mathbb{Z}_2])$, the $\pm$ and 2-forms are respectively $\lambda_{\pm} = H(\mathbb{Z}_{\pm})$ and $\lambda_2 = H(\mathbb{Z}_2)$. This follows readily from the definition and from the fact that the quotient-induced isomorphism $\Lambda^{\mathbb{Q}} \cong \frac{\Lambda^{\mathbb{Q}}}{(1-T)} \oplus \frac{\Lambda^{\mathbb{Q}}}{(1+T)}$ maps 1 to $([1],[1])$ and 0 to 0.

Example 4.6. We prove that the pullback of a direct sum is the direct sum of the pullbacks. Namely, we prove that if $(M,\lambda),(M/M_\mp,\lambda_\pm),(M_2,\lambda_2)$ and $(N,\kappa),(N/N_\mp,\kappa_\pm),(N_2,\kappa_2)$ form two pullback squares, then the following is a pullback:

$$\begin{array}{ccc} (M,\lambda)\oplus(N,\kappa) & \longrightarrow & (M/M_+,\lambda_-)\oplus(N/N_+,\kappa_-) \\ \downarrow & & \downarrow \\ (M/M_-,\lambda_+)\oplus(N/N_-,\kappa_+) & \longrightarrow & (M_2,\lambda_2)\oplus(N_2,\kappa_2). \end{array}$$

Here, we have used the canonical isomorphism $(M\oplus N)/(M\oplus N)_\pm \cong (M/M_\pm)\oplus(N/N_\pm)$. By Theorem 4.4, it suffices to prove that $(\lambda\oplus\kappa)_\pm=\lambda_\pm\oplus\kappa_\pm$. This follows from the fact that in this case the commutative diagram in Construction 4.2 splits as a direct sum of diagrams.

The output of Example 4.5 and Example 4.6 is that the stabilisation of a pullback is the pullback of the stabilisations (apply Example 4.6 with $N=H(\mathbb{Z}[\mathbb{Z}_2])$ and use Example 4.5 to compute $N_\pm$ and N_2). This is one of the main facts that we will use in the proof of Theorem 1.2.

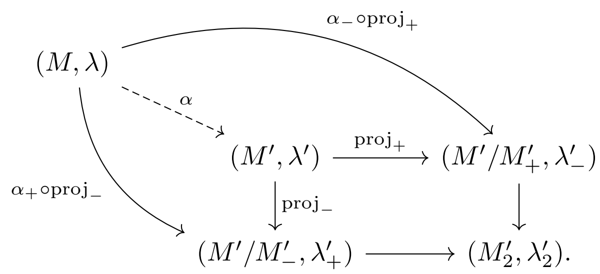
We prove a condition on how to achieve an isometry $(M,\lambda)\to(M',\lambda')$ of two pullback forms.

Proposition 4.7. *Let (M,λ) and (M',λ') be two pullback forms and fix isometries*

$$\alpha_-\colon (M/M_+,\lambda_-)\xrightarrow{\cong}(M'/M'_+,\lambda'_-) \qquad \alpha_+\colon (M/M_-,\lambda_+)\xrightarrow{\cong}(M/M'_-,\lambda'_+).$$

If α_+ and α_- agree on $M/(M_-,M_+)$, then there exists an isometry $\alpha\colon(M,\lambda)\to(M',\lambda')$.

Proof. Use the given isometries and apply the universal property of module pullbacks to obtain an isomorphism α fitting into the diagram



Since $\alpha_\pm$ are isometries, one can verify that the pushforward form $\alpha_*(\lambda):=\lambda(\alpha(-),\alpha(-))$ is a form on M' satisfying properties (2) and (3). Hence, by the second item of Theorem 4.4, $\alpha_*(\lambda)=\lambda'$, and so α is the desired isometry. $\square$

4.2. **Pullbacks and equivariant intersection forms.** The goal of this section is to apply the pullback construction to equivariant intersection forms of $\mathbb{Z}_2$-surface exteriors.

In what follows, recall that we denote the exterior of a $\mathbb{Z}_2$-surface $F\subset S^4$ by X_F. Note that since the universal cover $\widetilde{X}_F$ of X_F is simply-connected, the $\mathbb{Z}[\mathbb{Z}_2]$-module $H_2(\widetilde{X}_F)$ is torsion-free as an abelian group, as required by the pullback construction. Recall that the intersection form $Q_{\widetilde{X}_F}$ on $H_2(\widetilde{X}_F)$ descends to a non-degenerate symmetric bilinear form $Q^{nd}_{\widetilde{X}_F}$ on $H_2(\widetilde{X}_F)/\operatorname{rad}$.

Proposition 4.8. *Given a $\mathbb{Z}_2$-surface $F\subset S^4$ of non-orientable genus h, the pullback square associated to the hermitian form $(H,\lambda):=(H_2(\widetilde{X}_F),\lambda_{X_F})$ is*

$$\begin{array}{ccc} (H,\lambda) & \longrightarrow & (H/H_+,2Q^{nd}_{\widetilde{X}_F}) \\ \downarrow & & \downarrow \\ (H/H_-,0) & \longrightarrow & (H/(H_+,H_-),0). \end{array}$$

Here $H_+=\operatorname{rad}(\lambda)$ and, in addition, there is an isomorphism $H\cong\mathbb{Z}_-\oplus\mathbb{Z}[\mathbb{Z}_2]^{h-1}$ which induces isomorphisms

$$H/H_+\cong\mathbb{Z}_-^h,\qquad H/H_-\cong\mathbb{Z}_+^{h-1},\qquad \textit{and}\qquad H/(H_-,H_+)\cong\mathbb{Z}_2^{h-1}.$$

Proof. Consider the projections $\mathrm{proj}\colon H \to H/\mathrm{rad}$ and $\mathrm{proj}_\pm\colon H \to H/H_\pm$.

We assert that $\mathrm{rad} = H_+$ so that $\mathrm{proj}_+ = \mathrm{proj}$. First, the inclusion $\mathrm{rad} \subset H_+$ follows because $\mathrm{rad}(\lambda_{X_F}) = H_2(\partial \widetilde{X}_F) \cong \mathbb{Z}_+^{h-1}$; see e.g. [COP23, proof of Proposition 5.12]. We therefore obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{rad} & \longrightarrow & H & \xrightarrow{\mathrm{proj}} & H/\mathrm{rad} & \longrightarrow & 0 \\ & & \downarrow \subset & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & H_+ & \longrightarrow & H & \xrightarrow{\mathrm{proj}_+} & H/H_+ & \longrightarrow & 0. \end{array}$$

Since [COP23, Proposition 4.12] ensures that $H \cong \mathbb{Z}_- \oplus \mathbb{Z}[\mathbb{Z}_2]^{h-1}$, we get $H/H_+ \cong \mathbb{Z}_-^h$. Using that $H/\mathrm{rad} \cong \mathbb{Z}_-^h$ (thanks to [COP23, Proposition 5.12]), we deduce that the right hand vertical map is a quotient map from $\mathbb{Z}_-^h$ that doesn't change the isomorphism type, and hence the quotient is trivial i.e. $H_+ \setminus \mathrm{rad} = \emptyset$. Hence we have that $\mathrm{rad} = H_+$, concluding the proof of the assertion.

Claim 1. For $x, y \in H$, the following equality holds:

$$\lambda_{X_F}(x,y) = (1-T)Q^{nd}_{\widetilde{X}_F}(x_-, y_-).$$

Here, recall the notation $x_- := \mathrm{proj}_+(x)$ and $y_- := \mathrm{proj}_+(y)$.

Proof of Claim 1. Write $\lambda^{nd}_{X_F}$ for the $\mathbb{Z}[\mathbb{Z}_2]$-form induced by λ_{X_F} on H/rad. Using the expression for $\lambda^{nd}_{X_F}$ derived in [COP23, Proposition 5.12], we obtain

$$\lambda_{X_F}(x,y) = \lambda^{nd}_{X_F}(\mathrm{proj}(x), \mathrm{proj}(y)) = (1-T)Q^{nd}_{\widetilde{X}_F}(\mathrm{proj}(x), \mathrm{proj}(y)). \tag{5}$$

Combining the assertion with (5) implies that

$$\lambda_{X_F}(x,y) = (1-T)Q^{nd}_{\widetilde{X}_F}(\mathrm{proj}_+(x), \mathrm{proj}_+(y)) = (1-T)Q^{nd}_{\widetilde{X}_F}(x_-, y_-).$$

This concludes the proof of Claim 1. □

We now verify that the proposed $+$ and $-$ forms satisfy (2) and (3). The result will then follow from Theorem 4.4. For the first condition (rewritten as in (4)), this follows because we get

$$\begin{aligned} \frac{(1-T)(\lambda_{X_F})_-(x_-,y_-) + (1+T)(\lambda_{X_F})_+(x_+,y_+)}{2} &= \frac{(1-T)2Q^{nd}_{\widetilde{X}_F}(x_-,y_-) + 0}{2} \\ &= (1-T)Q^{nd}_{\widetilde{X}_F}(x_-,y_-). \end{aligned}$$

We now show (3) i.e. that $(H/\mathrm{rad}, 2Q^{nd}_{\widetilde{X}_F})$ reduces to $(H/(H_+, H_-), 0)$ modulo 2. This is clear because of the factor 2 in the form. The isomorphisms $H/H_- \cong \mathbb{Z}_+^{h-1}$ and $H/(H_-, H_+) \cong \mathbb{Z}_2^{h-1}$ follow from the fact that $H \cong \mathbb{Z}_- \oplus \mathbb{Z}[\mathbb{Z}_2]^{h-1}$; see [COP23, Proposition 4.12]. □

Convention 4.9. Given a $\mathbb{Z}_2$-surface $F \subset S^4$ of non-orientable genus h, from now on, we fix a $\mathbb{Z}[\mathbb{Z}_2]$-isomorphism $H_2(\widetilde{X}_F) \cong \mathbb{Z}_- \oplus \mathbb{Z}[\mathbb{Z}_2]^{h-1}$ as at the end of Proposition 4.8 to identify the therein pullback square with the pullback square

$$\begin{array}{ccc} (H, \lambda) & \longrightarrow & (\mathbb{Z}_-^h, 2Q^{nd}_{\widetilde{X}_F}) \\ \downarrow & & \downarrow \\ (\mathbb{Z}_+^{h-1}, 0) & \longrightarrow & (\mathbb{Z}_2^{h-1}, 0). \end{array}$$

The bottom horizontal map is surjective (as is the right vertical map), but we do not claim that the bottom horizontal map is reduction mod two. It can, however, be factored as an isomorphism $\mathbb{Z}_+^{h-1} \to \mathbb{Z}_+^{h-1}$ postcomposed with reduction mod two. This follows since the map $GL_{h-1}(\mathbb{Z}) \to GL_{h-1}(\mathbb{Z}_2)$ is surjective. We record that Proposition 4.7 applies also to this identified square by post/pre-composing all relevant maps with the required identifications.

We conclude by recording a technical point related to the pullback construction.

Remark 4.10. Contrary to the situation described in Example 4.5, we emphasise that in Proposition 4.8 the map $H_2(\widetilde{X}_F)/\operatorname{rad} \cong \mathbb{Z}_-^h \to \mathbb{Z}_2^{h-1}$ should not be thought of as reduction mod 2: notice the difference in rank. This is already visible when $h = 1$; in this case, the pullback square takes the form

$$\begin{array}{ccc} \mathbb{Z}_- & \to & \mathbb{Z}_- \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0. \end{array}$$

Here we see that the map $\mathbb{Z}_- \to 0$ is not reduction mod 2.

5. Constructing non-smoothable surfaces

This section is devoted to the proof of Theorems 1.2 and 1.6. We focus on the proof of Theorem 1.2 and then afterwards give the small modification necessary for Theorem 1.6.

We briefly outline the strategy of the proof as well as the organisation of the section. Start by using Examples 4.5 and 4.6 to consider the once stabilised pullback square associated to the unknotted surface $U_h \subset S^4$ with non-orientable genus h and (extremal) normal Euler number $-2h$ (recall Proposition 4.8). Setting $u := Q_{\Sigma_2(U_h)}$ for the diagonal definite non-singular form of rank h, this pullback takes the following form:

$$\begin{array}{ccc} \overbrace{(H_2(\widetilde{X}_{U_h}), \lambda_{X_{U_h}})}^{:=(M_{U_h},\lambda_{U_h})} \oplus H(\mathbb{Z}[\mathbb{Z}_2]) & \to & (\mathbb{Z}_-^h, 2u_{ext}^{nd}) \oplus H(\mathbb{Z}_-) \\ \downarrow & & \downarrow \\ (\mathbb{Z}_+^{h-1}, 0) \oplus H(\mathbb{Z}_+) & \longrightarrow & (\mathbb{Z}_2^{h-1}, 0) \oplus H(\mathbb{Z}_2). \end{array} \tag{6}$$

The first summands of the bottom horizontal map and the right vertical map are defined to be the ones from Convention 4.9. On the hyperbolic summands, the horizontal (resp. vertical) maps are induced by quotienting by $1 + T$ (resp. $1 - T$) as in Example 4.5.

Now fix an odd definite non-singular symmetric bilinear form $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$. This is the form that we wish to realise as the intersection form of the double branched cover of a $\mathbb{Z}_2$-surface. Use Examples 4.5 and 4.6 to form the once stabilised pullback square associated to b (which currently has no surface attached to it):

$$\begin{array}{ccc} (M, \lambda_b) \oplus H(\mathbb{Z}[\mathbb{Z}_2]) & \to & (\mathbb{Z}_-^h, 2b_{ext}^{nd}) \oplus H(\mathbb{Z}_-) \\ \downarrow & & \downarrow \\ (\mathbb{Z}_+^{h-1}, 0) \oplus H(\mathbb{Z}_+) & \longrightarrow & (\mathbb{Z}_2^{h-1}, 0) \oplus H(\mathbb{Z}_2). \end{array} \tag{7}$$

The bottom horizontal map and the right vertical map are defined to be the same as in (6). The main technical step in the proof of Theorem 1.2 consists of stably realising λ_b. This is achieved in Proposition 5.3 but here we state the main steps.

In order to stably realise λ_b, Proposition 5.1 first builds an isometry

$$\alpha_-\colon (\mathbb{Z}_-^h, 2u_{ext}^{nd}) \oplus H(\mathbb{Z}_-) \to (\mathbb{Z}_-^h, 2b_{ext}^{nd}) \oplus H(\mathbb{Z}_-).$$

Proposition 5.2 then constructs an isometry

$$\alpha_+\colon (\mathbb{Z}_+^{h-1}, 0) \oplus H(\mathbb{Z}_+) \to (\mathbb{Z}_+^{h-1}, 0) \oplus H(\mathbb{Z}_+)$$

that induces the same isometry as α_- on

$$(\mathbb{Z}_2^{h-1}, 0) \oplus H(\mathbb{Z}_2) \to (\mathbb{Z}_2^{h-1}, 0) \oplus H(\mathbb{Z}_2).$$

The stable realisability of λ_b then follows from the work of Hambleton-Riehm (Proposition 4.7). The construction of α_-, α_+ and the proof of stable realisation are carried out in Sections 5.1-5.3.

Once λ_b is stably realised, the remaining steps are carried out in Section 5.4. In a nutshell, Proposition 2.1 ensures that λ_b is in fact realised and it is then relatively routine to deduce that the form in fact arises as the equivariant intersection form of the exterior of a $\mathbb{Z}_2$-surface $F \subset S^4$.

We then show that $Q_{\Sigma_2(F)} \cong b$. This step is somewhat technical, since, due to the nature of our construction, we can only a priori deduce that $Q_{\Sigma_2(F)}$ and b share an index two subform.

5.1. **Building the stable isometry α_- of the minus forms.** Recall that, as above, we write $U_h \subset S^4$ for the unknotted surface of non-orientable genus h with (extremal) normal Euler number $-2h$ and set $u := Q_{\Sigma_2(U_h)}$. The goal of this section is to prove the following result.

Proposition 5.1. *If $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ is an odd definite non-singular symmetric bilinear form, then there exists a $\mathbb{Z}[\mathbb{Z}_2]$-isometry*

$$\alpha_- \colon (\mathbb{Z}^h_-, 2u^{nd}_{ext}) \oplus H(\mathbb{Z}_-) \to (\mathbb{Z}^h_-, 2b^{nd}_{ext}) \oplus H(\mathbb{Z}_-).$$

Proof. By the classification of non-singular indefinite symmetric bilinear forms (see e.g. [Ser78]), the $\mathbb{Z}$-forms b and $u = (1)^{\oplus h}$ become isometric after a single stabilisation (these stabilisations are indefinite, have the same rank, parity, and signature).

By Proposition 3.3, the forms b^{nd}_{ext} and u^{nd}_{ext} are stably isometric. However, we need a stable isometry of the forms $2b^{nd}_{ext}$ and $2u^{nd}_{ext}$. By the classification of non-degenerate forms due to Nikulin [Nik79, Corollary 1.13.4], $2b^{nd}_{ext}$ and $2u^{nd}_{ext}$ are stably isometric (after one stabilisation) if and only if they have the same rank, signature and quadratic boundary linking form (for a definition see e.g. [COP23, Section 7.4]). The first two follow readily from the fact that b^{nd}_{ext} and u^{nd}_{ext} are stably isometric. The third also follows from this, but the argument is more involved.

Claim 2. The forms $2b^{nd}_{ext}$ and $2u^{nd}_{ext}$ have isometric quadratic boundary linking forms.

Proof of Claim 2. This claim can be proved using the definition of the quadratic boundary linking form mentioned e.g. in [COP23, Definition 7.26] but we choose to work with matrices in order to avoid recalling background notions related to quadratic forms.

Choose matrices A_b, A_u to represent the symmetric forms $b^{nd}_{ext}, u^{nd}_{ext}$ and Q_b, Q_u to represent the underlying quadratic forms, respectively, so that $A_u = Q_u + Q_u^T$ and $A_b = Q_b + Q_b^T$. Use $q_u\colon \mathbb{Z}^h/A_u \to \mathbb{Q}/\mathbb{Z}$ and $q_b\colon \mathbb{Z}^h/A_b \to \mathbb{Q}/\mathbb{Z}$ to denote the quadratic boundary linking forms for b^{nd}_{ext} and u^{nd}_{ext}, respectively. Taking inverses in $\mathbb{Q}$, as noted in [COP23, Remark 7.27], the quadratic linking forms may be computed as

$$q_b(\pi(z)) = z^T(A_b^{-1})^T Q_u A_b^{-1} z \quad \text{ and } \quad q_u(\pi(z)) = z^T(A_u^{-1})^T Q_u A_u^{-1} z.$$

Here $\pi\colon \mathbb{Z}^h \to \mathbb{Z}^h/A_b$ denotes the canonical projection and similarly for A_u.

Similarly, the symmetric forms $2b^{nd}_{ext}$, $2u^{nd}_{ext}$ and their underlying quadratic forms may then be represented by the matrices $2A_b, 2A_u, 2Q_b, 2Q_u$, respectively. This way, the corresponding quadratic linking forms $q_{2u}\colon \mathbb{Z}^h/2A_u \to \mathbb{Q}/\mathbb{Z}$ and $q_{2b}\colon \mathbb{Z}^h/2A_b \to \mathbb{Q}/\mathbb{Z}$ for $2b^{nd}_{ext}$ and $2u^{nd}_{ext}$ are given by

$$q_{2b}(\pi_2(z)) = z^T((2A_b)^{-1})^T 2Q_u(2A_b)^{-1} z \quad \text{ and } \quad q_{2u}(\pi_2(z)) = z^T((2A_u)^{-1})^T 2Q_u(2A_u^{-1}) z.$$

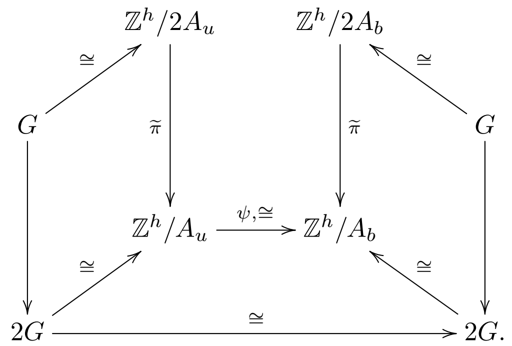
Here $\pi_2\colon \mathbb{Z}^h \to \mathbb{Z}^h/2A_b$ denotes the canonical projection and similarly for $2A_u$.

Since $b^{nd}_{ext} \oplus H$ and $u^{nd}_{ext} \oplus H$ are even and stably isometric, so are their underlying quadratic forms. This implies that their boundary quadratic linking forms are isometric. Stabilising by a hyperbolic does not affect these linking forms, so there exists a quadratic linking form isometry $\psi\colon (\mathbb{Z}^h/A_u, q_u) \to (\mathbb{Z}^h/A_b, q_b)$.

We assert that the isomorphism $\psi\colon \mathbb{Z}^h/A_u \to \mathbb{Z}^h/A_b$ underlying this isometry lifts to an isomorphism $\widetilde{\psi}\colon \mathbb{Z}^h/2A_u \to \mathbb{Z}^h/2A_u$ that makes the following diagram commute:

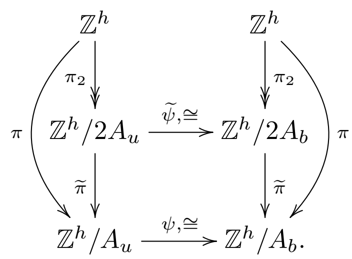
$$\begin{array}{ccc} \mathbb{Z}^h/2A_u & \overset{\widetilde{\psi},\cong}{\dashrightarrow} & \mathbb{Z}^h/2A_b \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \widetilde{\pi}} \\ \mathbb{Z}^h/A_u & \xrightarrow{\psi,\cong} & \mathbb{Z}^h/A_b. \end{array}$$

Here $\widetilde{\pi}$ denotes the relevant projection maps. Recall that A_u is congruent to the matrix displayed in (1). Using row and column operations, one can show that this latter matrix (and therefore A_u) has Smith normal form $D = \operatorname{diag}(1, \ldots, 1, 4)$ if h is odd and to $D = \operatorname{diag}(1, \ldots, 1, 2, 2)$ if h is even. Set $G := \operatorname{coker}(2D)$ for brevity from which it follows that $2G \cong \operatorname{coker}(D) \in \{\mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$. Putting A_b into Smith normal form and then performing the same row and column operations on $2A_u$ and $2A_b$ respectively leads to the following commutative diagram (in which the diagonal arrows arise from said operations, the maps $G \to 2G$ are the canonical ones, and the bottom horizontal map is the unique isomorphism that makes the diagram commute):



The assertion therefore reduces to proving that $\operatorname{Aut}(G) \to \operatorname{Aut}(2G)$ is surjective. Since $2G$ is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, this is clear: in the first case because $\operatorname{Aut}(\mathbb{Z}_4) = \{\pm 1\}$ and in the second because $GL_2(\mathbb{Z}_4) \to GL_2(\mathbb{Z}_2)$ is surjective. This concludes the proof of the assertion.

It remains to prove that the isomorphism $\widetilde{\psi}$ from the assertion is an isometry of quadratic linking forms. Consider the following commutative diagram:



Choose $y \in \mathbb{Z}^h$ so that $\pi_2(y) = \widetilde{\psi}(\pi_2(z))$. It follows that $\pi(y) = \psi(\pi(z))$ so that y can be used to calculate both $q_{2b}(\widetilde{\psi}(\pi_2(z)))$ and $q_b(\psi(\pi(z)))$. We deduce that

$$\begin{aligned} q_{2u}(\pi_2(z)) &= \frac{1}{2} z^T (A_u^{-1})^T Q_u A_u^{-1} z = \frac{1}{2} q_u(\pi(z)) = \frac{1}{2} q_b(\psi(\pi(z))) \\ &= \frac{1}{2} y^T (A_b^{-1})^T Q_b A_b^{-1} y = q_{2b}(\widetilde{\psi}(\pi_2(z))). \end{aligned}$$

This concludes the proof of Claim 2. □

We now appeal to the aforementioned classification of even non-degenerate symmetric bilinear forms due to Nikulin [Nik79, Corollary 1.13.4] which says that $2b^{nd}_{ext}$ and $2u^{nd}_{ext}$ are stably isometric (after one stabilisation) since they have the same rank, signature, and quadratic boundary linking form. The output is the required isometry

$$\alpha_- : (\mathbb{Z}^h_-, 2u^{nd}_{ext}) \oplus H(\mathbb{Z}_-) \xrightarrow{\cong} (\mathbb{Z}^h_-, 2b^{nd}_{ext}) \oplus H(\mathbb{Z}_-).$$

Here recall that α_- is automatically a $\mathbb{Z}[\mathbb{Z}_2]$-isometry because the modules are endowed with the $\mathbb{Z}_2$-action induced by $Tx = -x$. This concludes the proof of the proposition. □

5.2. **Building the stable isometry α_+ of the plus forms.** Proposition 5.1 ensures the existence of an isometry

$$\alpha_- : (\mathbb{Z}^h_-, 2u^{nd}_{ext}) \oplus H(\mathbb{Z}_-) \xrightarrow{\cong} (\mathbb{Z}^h_-, 2b^{nd}_{ext}) \oplus H(\mathbb{Z}_-).$$

The goal of this section is to build a compatible isometry α_+.

Proposition 5.2. *Fix an odd definite non-singular symmetric bilinear form* $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ *and an isometry*

$$\alpha_-\colon (\mathbb{Z}^h_-, 2u^{nd}_{ext}) \oplus H(\mathbb{Z}_-) \xrightarrow{\cong} (\mathbb{Z}^h_-, 2b^{nd}_{ext}) \oplus H(\mathbb{Z}_-).$$

There exists an isometry

$$\alpha_+\colon (\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+) \xrightarrow{\cong} (\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+)$$

that induces the same isometry as α_- *on*

$$(\mathbb{Z}^{h-1}_2, 0) \oplus H(\mathbb{Z}_2) \xrightarrow{\cong} (\mathbb{Z}^{h-1}_2, 0) \oplus H(\mathbb{Z}_2).$$

Proof. We view this as a two stage lifting problem. Consider the diagram

$$\begin{array}{ccc}
(\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+) & \overset{\alpha_+}{\dashrightarrow} & (\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+) \\
\cong \downarrow \beta\oplus\mathrm{id} & & \cong \downarrow \beta\oplus\mathrm{id} \\
(\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+) & \overset{\widetilde{\alpha_2}}{\dashrightarrow} & (\mathbb{Z}^{h-1}_+, 0) \oplus H(\mathbb{Z}_+) \\
\downarrow \mathrm{red}_2 \oplus \mathrm{red}_2 & & \downarrow \mathrm{red}_2 \oplus \mathrm{red}_2 \\
(\mathbb{Z}^{h-1}_2, 0) \oplus H(\mathbb{Z}_2) & \xrightarrow{\alpha_2, \cong} & (\mathbb{Z}^{h-1}_2, 0) \oplus H(\mathbb{Z}_2),
\end{array}$$

where α_2 is the isometry determined by α_-. The columns correspond to factorisations of the bottom rows in (5) and (6), respectively. That they factor this way follows from Convention 4.9 for the first summand and Example 4.5 for the second summand.

We wish to first lift α_2 to an isometry $\widetilde{\alpha_2}$. The underlying map α_2 is determined by a matrix with entries in $\mathbb{Z}_2$ that we write as the $(h-1) \times (h-1)$ block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the matrix D is a 2×2 block. A rapid calculation using that α_2 is an isometry is seen to imply that D represents an isometry $H(\mathbb{Z}_2) \to H(\mathbb{Z}_2)$. There are only six such matrices (in fact $GL_2(\mathbb{Z}_2) \cong D_3$), namely

$$P_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, P_3 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, P_4 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, P_5 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, P_6 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and only the first two lift to isometries $H(\mathbb{Z}_+) \to H(\mathbb{Z}_+)$ (this fact can readily be verified). Another rapid calculation also shows that since α_2 is an isometry and D is invertible, C must be the zero matrix: $C = \mathbf{0}$. Since α_2 is an isomorphism, it follows that A is invertible.

Claim 3. The matrix D is either P_1 or P_2.

Proof of Claim 3. For brevity, we write $(\lambda_u)^{stab}_- := 2u^{nd}_{ext} \oplus H(\mathbb{Z}_-)$ and $(\lambda_b)^{stab}_- := 2b^{nd}_{ext} \oplus H(\mathbb{Z}_-)$. From its definition we deduce that b^{nd}_{ext} has diagonal entries even, and so $2b^{nd}_{ext}$ has diagonal entries divisible by four and off-diagonal entries even. Use e, f to denote the canonical basis of the hyperbolic summand in $\mathbb{Z}^h_- \oplus H(\mathbb{Z})$.

Since the maps $H(\mathbb{Z}_-) \to H(\mathbb{Z}_2)$ in (6) and (7) are given by reduction mod 2, the map α_- can be represented by a 2×2 block matrix whose lower right corner is given by a 2×2 matrix of the form $\left(\begin{smallmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{smallmatrix}\right)$ that reduces mod 2 to D.

With this notation, $\alpha_-(e) = (x \ d_{11} \ d_{21})$ for some $x \in \mathbb{Z}^h$. Note also that $(\lambda_u)^{stab}_-(e, e) = 0$. Since α_- is an isometry, it follows that

$$\begin{aligned}
0 &= (\lambda_b)^{stab}_-(\alpha_-(e), \alpha_-(e)) \\
&= x^T(2b^{nd}_{ext})x + \begin{pmatrix} d_{11} & d_{21} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} \\
&= x^T(2b^{nd}_{ext})x + 2d_{11}d_{21}.
\end{aligned}$$

Since $2b^{nd}_{ext}$ has diagonal entries that are multiples of 4 and offdiagonal entries that are even, we deduce that $x^T(2b^{nd}_{ext})x = 0 \mod 4$. It follows that $2d_{11}d_{21} = 0 \mod 4$ which forces $d_{11}d_{21}$ to be even. Repeating the argument with f instead of e shows that $d_{12}d_{22}$ must also be even. This concludes the proof of Claim 3. □

Using Claim 3 we know that the matrix D lifts to a isometry of $H(\mathbb{Z}_+)$ which we denote by D'. For the matrices A and B we are free to pick any integral lifts A', B'; they play no role in the resulting integral matrix being an isometry. Since A is invertible and $GL_{h-1}(\mathbb{Z}) \to GL_{h-1}(\mathbb{Z}_2)$ is surjective, we can assume that A' is invertible. It follows that the following matrix is also invertible:

$$\widetilde{\alpha_2} := \begin{pmatrix} A' & B' \\ \mathbf{0} & D' \end{pmatrix}.$$

Since D' is an isometry of $H(\mathbb{Z}_+)$, so is $\widetilde{\alpha_2}$. Finally, we define

$$\alpha_+ := (\beta \oplus \mathrm{id})^{-1} \circ \widetilde{\alpha_2} \circ (\beta \oplus \mathrm{id}).$$

Since β and $\widetilde{\alpha_2}$ are isometries, it follows that α_+ is the required isometry. □

5.3. **Stably realising λ_b.** Recall that $U_h \subset S^4$ denotes the unknotted surface with non-orientable genus h and normal Euler number $-2h$, and that we set $(M_{U_h}, \lambda_{U_h}) := (H_2(\widetilde{X}_{U_h}), \lambda_{X_{U_h}})$. Combining the previous propositions with the work of Hambleton-Riehm yields the following result.

Proposition 5.3. *If $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ is an odd definite non-singular symmetric bilinear form, then there exists a $\mathbb{Z}[\mathbb{Z}_2]$-isometry*

$$\alpha\colon (M_{U_h}, \lambda_{U_h}) \oplus H(\mathbb{Z}[\mathbb{Z}_2]) \xrightarrow{\cong} (M, \lambda_b) \oplus H(\mathbb{Z}[\mathbb{Z}_2]).$$

In particular, the form λ_b is stably realisable.

Proof. Proposition 5.1 and Proposition 5.2 produce a compatible pair of isometries α_- and α_+. The existence of α now follows from Proposition 4.7. The hermitian form λ_b is stably realisable because $(M_{U_h}, \lambda_{U_h}) \oplus H(\mathbb{Z}[\mathbb{Z}_2])$ is isometric to the $\mathbb{Z}[\mathbb{Z}_2]$-intersection form of $X_{U_h} \# S^2 \times S^2$. □

5.4. **Realising the form by a surface.** We now prove the previously mentioned refinement of our main theorem, which we restate for the reader.

Theorem 1.2. *If $b\colon \mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}$ is an odd definite non-singular symmetric bilinear form of rank $h \not\equiv 4 \mod 8$, then there exists a non-orientable $\mathbb{Z}_2$-surface $F \subset S^4$ such that $Q_{\Sigma_2(F)} \cong b$.*

Proof. The output of Proposition 5.3 is that λ_b is stably realisable, and hence by applying Proposition 2.1, we obtain that λ_b is realisable by a $\mathbb{Z}_2$-manifold X' with boundary $Y = \partial X_{U_h}$. Here, for the coefficient system we use the inclusion induced map $\varphi\colon \pi_1(\partial X_{U_h}) \to \pi_1(X_{U_h}) \cong \mathbb{Z}_2$.

Since $Y = \partial X_{U_h}$ we then cap off X' with the normal bundle of U_h in S^4 (which has Euler number $-2h$); denote the outcome by $\widehat{X}'$. The choice of φ ensures that $\widehat{X}'$ is a closed simply-connected 4-manifold (this follows by a simple Seifert-Van Kampen argument). By construction, it contains a locally flat embedded $\mathbb{Z}_2$-surface, namely the 0-section of the aforementioned disc bundle, which we denote by F.

Claim 4. The 4-manifold $\widehat{X}'$ satisfies $b_2(\widehat{X}') = 0$.

Proof of Claim 4. Since the 4-manifold X' has $\pi_1(X') \cong \mathbb{Z}_2$ and $\pi_1(Y) \to \pi_1(X') \cong \mathbb{Z}_2$ is surjective, the argument from [COP23, Proof of Proposition 4.12] shows that $H_2(\widetilde{X'}) \cong \mathbb{Z}_- \oplus \mathbb{Z}[\mathbb{Z}_2]^{h-1}$. Since $\widetilde{X'}$ double covers X', the multiplicativity of the Euler characteristic gives $b_2(X') = h - 1$:

$$2(1 + b_2(X')) = 2\chi(X') = \chi(\widetilde{X'}) = 1 + b_2(\widetilde{X'}) = 2 + 2(h - 1).$$

The additivity of the Euler characteristic then shows that $b_2(\widehat{X}') = 0$:

$$2 + b_2(\widehat{X}') = \chi(\widehat{X}') = \chi(X') + \chi(F) = (1 + b_2(X')) + (2 - h) = h + (2 - h) = 2.$$

This concludes the proof of Claim 4. □

Thus $\widehat{X}'$ is a closed simply-connected with vanishing b_2. It is therefore homeomorphic to S^4 (by Freedman's work [Fre82]) and contains a $\mathbb{Z}_2$-surface F with non-orientable genus h, Euler number $-2h$, and exterior having equivariant intersection form isometric to λ_b.

It remains to prove that $Q_{\Sigma_2(F)} \cong b$. Since $\lambda_{X_F} \cong \lambda_b$ we deduce that $(\lambda_{X_F})_- \cong 2Q^{nd}_{\widetilde{X}_F}$ is isometric to $(\lambda_b)_- \cong 2b^{nd}_{ext}$. Cancelling the 2s, which is possible since these forms are $\mathbb{Z}$-valued, implies that $b^{nd}_{ext} \cong Q^{nd}_{\widetilde{X}_F}$. Thus b and $Q_{\Sigma_2(F)}$ share $b^{nd}_{ext} \cong Q^{nd}_{\widetilde{X}_F}$ as an index 2 subform. Since $h \not\equiv 4$ mod 8, results of Chenevier [Che25, Proposition-Definition 3.2(i) and Proposition 3.8 (i)] (see Proposition A.5 for a translation of Chenevier's work to our setting) imply that $Q_{\Sigma_2(F)} \cong b$. □

We now give the proof of Theorem 1.6, which we restate for the reader.

Theorem 1.6. *For every $h \geq 9$ and every $e \in \{-2h, -2h+4, \ldots, 2h-4, 2h\}$, there exists a knotted $\mathbb{Z}_2$-surface of non-orientable genus h and Euler number e if and only if $|e| = 2h$.*

Proof. If $|e| \neq 2h$, then every $\mathbb{Z}_2$-surface of non-orientable genus h and Euler number e is unknotted [COP23, Theorem A]. We therefore focus on the converse. Set $b = E_8 \oplus (1)^{\oplus k}$ and run the proof of Theorem 1.2 up until the last sentence. This constructs a $\mathbb{Z}_2$-surface $F \subset S^4$ such that $b^{nd}_{ext} \cong Q^{nd}_{\widetilde{X}_F}$. We use a root-counting argument to distinguish $Q^{nd}_{\widetilde{X}_F} \cong E_8 \oplus \left((1)^{\oplus k}\right)^{nd}_{ext}$ from $\left((1)^{\oplus(8+k)}\right)^{nd}_{ext}$ and hence prove that $Q_{\Sigma_2(F)} \not\cong (1)^{\oplus(8+k)}$, which implies that F is knotted. The number of roots (vectors of length two) in b^{nd}_{ext} is $240 + 4\binom{k}{2}$ whereas for $\left((1)^{\oplus(8+k)}\right)^{nd}_{ext}$ it is $4\binom{k+8}{2}$; see Proposition B.1 for this calculation. These counts differ for $k \geq 1$ except when $k = 4$. When $k = 4$, the result follows from [Che25, Section 1.4]. In more detail, if there is an isometry $E_8 \oplus \left((1)^{\oplus 4}\right)^{nd}_{ext} \cong \left((1)^{\oplus 12}\right)^{nd}_{ext}$, then Lemma A.3 implies that either $E_8 \oplus (1)^{\oplus 4} \cong (1)^{\oplus 12}$ or $E_8 \oplus (1)^{\oplus 4}$ is isometric to a neighbour of $(1)^{\oplus 12}$. Neither option is possible, for the former this is clear whereas for the latter, [Che25, Section 1.4] states that the only neighbour of $(1)^{\oplus 12}$ is Γ_{12}. □

Appendix A. Neighbouring lattices

The purpose of this section is to translate elements of the work of Chenevier [Che25] on Kneser neighbours to our setting for the purpose of applying them at the end of the proof of Theorem 1.2. This translation is purely formal.

Definition A.1. A *lattice* of rank h is a subgroup $L \subset \mathbb{R}^h$ generated by a basis of $\mathbb{R}^h$. Two lattices L_1, L_2 are *isometric* if there is an isomorphism $\mathbb{R}^h \to \mathbb{R}^h$ preserving the standard inner product that sends L_1 to L_2.

Given an unimodular lattice L, we pick a generating basis (v_i) and take the *Gram matrix* M_L whose entries are $v_i \cdot v_j$ where $\cdot$ denotes the standard inner product. This matrix is only well-defined up to conjugacy, but the underlying non-singular symmetric bilinear form is well-defined and we call it b_L. This procedure has an inverse. Given a positive definite non-singular symmetric bilinear form b, use the standard basis of $\mathbb{R}^h$ to write b as a symmetric, positive definite matrix M. For every such matrix, there exists a decomposition $M = B^T B$ and the columns of B can now be taken to be a new basis for $\mathbb{R}^h$ such that M is the Gram matrix of $\cdot$ with respect to this basis. Taking the span of this basis gives a lattice L_b. It is not hard to see that $b_{L_b} = b$ and $L_{b_L} = L$. Similarly, one can check that the concepts of isometry for lattices and forms coincide.

We recall the definition of a neighbouring lattice due to Kneser [Kne57]; see also e.g. [Voi23] and [Che25, Section 3.1]).

Definition A.2. Given a unimodular lattice L and an integer $d \geq 1$, a unimodular lattice N is a *d-neighbour* of L if $L/(L \cap N) \cong \mathbb{Z}_d$, i.e. if N intersects L precisely in an index d sublattice. When $d = 2$, we refer to N as a *neighbour* of L.

It is known that L is a d-neighbour of N if and only if N is a d-neighbour of L; see e.g. [Che25, Section 3.1]. Restricting to the case $d = 2$, we discuss the exterior non-degenerate form in the setting of lattices and relate it to neighbours.

Given a unimodular lattice L and a d-primitive vector $x \in L$ (meaning that $[x] \in L/dL$ generates a subgroup of order d [Che25, Section 3.1]), Chenevier defines

$$M_d(L;x) := \{m \in L \mid m \cdot x \equiv 0 \mod d\}.$$

When L is odd, characteristic vectors $\xi \in L$ are 2-primitive, in which case $M_2(L;\xi)$ does not depend on the choice of the characteristic vector since

$$M_2(L) := M_2(L;\xi) = \{m \in L \mid m \cdot \xi \equiv 0 \mod 2\} = \{m \in L \mid m \cdot m \equiv 0 \mod 2\}.$$

This allows us to state the following.

Lemma A.3. *Let b and c be rank h odd definite non-singular symmetric bilinear forms. If there is an isometry $b^{nd}_{ext} \cong c^{nd}_{ext}$, then there exists an isometry $M_2(L_b) \cong M_2(L_c)$ and hence either L_b is isometric to a neighbour of L_c sharing $M_2(L_c)$, or $L_b \cong L_c$.*

Proof. The first statement follows from the fact that $L_{b^{nd}_{ext}} = M_2(L_b)$ which is immediately clear from the definitions. Let $\psi\colon \mathbb{Z}^h \to \mathbb{Z}^h$ denote the isometry. Then $\psi(L_b) \cap L_c$ contains $M_2(L_c)$ and since the intersection must be a sublattice, either $\psi(L_b) \cap L_c = L_c$ or $\psi(L_b) \cap L_c = M_2(L_c)$. In the latter case, then $\psi(L_b)$ and L_c are neighbours since $M_2(L_c)$ is index two in L_c. In the former case then $\psi(L_b) = L_c$ and so ψ was actually an isometry. □

Whereas the number of d-neighbours of a lattice L may be large [Che25, Corollary 3.6], the number of d-neighbours N with $N \cap L = M_d(L;x)$ for a fixed d-primitive vector $x \in L$ is typically much smaller [Che25, Proposition 3.2]. We will state a particular case of this result in the case where $d = 2$ (so that $e = 2$ in Chenevier's notation). In this setting, the answer depends on $x \cdot x \mod 8$ but since we are only interested in the case where $x = \xi$ is characteristic, we will formulate the results using the rank of the lattice instead: recall that for unimodular lattices, $h \equiv \xi \cdot \xi \mod 8$; see e.g. [GS99, Lemma 1.2.20].

We now state the main result we want to use which is a combination of [Che25, Proposition-Definition 3.2 (i)] and [Che25, Proposition 3.8 (i)].

Proposition A.4. *Let L be an odd integral unimodular lattice of rank h.*

(1) If $h \not\equiv 0, 4 \mod 8$, then L has no neighbours sharing the index two sublattice $M_2(L)$.
(2) If $h \equiv 0 \mod 8$, then L has precisely two neighbours sharing the index two sublattice $M_2(L)$, both of which are even.
(3) If $h \equiv 4 \mod 8$, then L has precisely two neighbours sharing the index two sublattice $M_2(L)$, both of which are odd.

Proof. Consider [Che25, Proposition-Definition 3.2 (i)] with $d = 2 = e$ and $x = \xi$ characteristic so that $M = M_2(L) = M_2(L;\xi)$. The first item of this result states the number of neighbours N with $L \cap N = M_2(L)$ is 2 if $\xi \cdot \xi = 0 \mod 4$ and 0 otherwise.

This directly implies the first item of the proposition, so we assume that $h \equiv 0 \mod 4$ and focus on the next two items. As mentioned above, in these cases L admits two neigbhours N with with $L \cap N = M_2(L)$. In [Che25, Remark 3.3], Chenevier describes these neighbours explicitly and denotes them by $N_2(L;\xi,0)$ and $N_2(L;\xi,1)$. Chenevier then shows that these neighbours are even if and only if $\xi \cdot \xi \equiv 0 \mod 8$ [Che25, Proposition 3.8 (i)]. This directly implies the last two items of the proposition. □

We give the translation to forms in the guise that we utilised in Section 5.

Proposition A.5. *Let b, c be odd definite non-singular symmetric bilinear forms of rank h and assume $b^{nd}_{ext} \cong c^{nd}_{ext}$. If $h \not\equiv 4 \mod 8$ then $b \cong c$.*

Proof. Apply Lemma A.3 to deduce that L_b is isometric to a lattice L such that either $L = L_c$, or L and L_c are neighbours sharing the index two sublattice $M_2(L_c)$. In the former case $L_b \cong L_c$, so $b \cong c$, as required. We therefore assume that L and L_c are neighbours over $M_2(L_c)$.

If $h \not\equiv 0, 4 \mod 8$, then by (1) of Proposition A.4 we have that L and L_c cannot be neighbours over $M_2(L_c)$, hence $L = L_c$. We again deduce that $b \cong c$.

If $h \equiv 0 \mod 8$, then by (2) of Proposition A.4 we have that all neighbours of L sharing the sublattice $M_2(L_c)$ are even, and hence L and L_c are not neighbours. This gives $L = L_c$ and so we deduce that $b \cong c$. □

Appendix B. Root-counting

This brief appendix details the count referenced at the end of the proof of Theorem 1.6. Recall that vectors of length two in a form are known as its *roots*. Let u_k denote the diagonal, definite form of rank k.

Proposition B.1. *The forms $E_8 \oplus (u_k)^{nd}_{ext}$ and $(u_{k+8})^{nd}_{ext}$ respectively admit $240+4\binom{k}{2}$ and $4\binom{k+8}{2}$ roots.*

Proof. Since these forms are positive definite and even, the number of roots is the sum of the number of roots in each direct summand. The E_8 form is well-known to have 240 roots. We now focus on deriving a formula for the number of roots of $(u_k)^{nd}_{ext}$. Denote the matrix representing this form by B (see the third item of Example 3.2). The equation $x^T Bx = 2$ can be written as

$$\begin{aligned}
2 &= x^T Bx \\
&= 4x_1^2 + 2\sum_{i=2}^{k} x_i^2 + 4\sum_{i=2}^{k} x_1 x_i + 2\sum_{\{2\le i\neq j\le k\}} x_i x_j \\
&= 4x_1^2 + 4x_1\sum_{i=2}^{k} x_i + \left(\sum_{i=2}^{k} x_i\right)^2 + \sum_{i=2}^{k} x_i^2 \\
&= \left(2x_1 + \sum_{i=2}^{k} x_i\right)^2 + \sum_{i=2}^{k} x_i^2 \\
&= X_1^2 + \sum_{i=2}^{k} x_i^2.
\end{aligned}$$

The set of integers $(X_1, x_2, \dots, x_k)$ satisfying this equation corresponds bijectively to the set of integers $(x_1, x_2, \dots, x_k)$ satisfying this equation. Indeed, if $(X_1, x_2, \dots, x_k)$ satisfies $X_1^2 + \sum_{i=2}^{k} x_i^2 = 2$, it follows that $X_1 - \sum_{i=2}^{k} x_i$ is even and therefore x_1 is uniquely determined by the equation $X_1 = 2x_1 + \sum_{i=2}^{k} x_i$.

Finally, for a sum of k integer squares to equal 2, exactly two of the terms must be ± 1 and the remaining must be 0. There are $\binom{k}{2} 2 \cdot 2 = 4\binom{k}{2}$ many such choices. □

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