

# STABLY EXOTIC KNOTTED SURFACES UP TO EQUIVALENCE

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**ABSTRACT.** In this short note we show the existence of stably exotic knotted surfaces in 4-manifolds in the sense that the embedded surfaces are topologically equivalent but are not smoothly equivalent even after any number of internal and external stabilisations.

Two embeddings  $s, s'$  of a surface  $\Sigma$  in a 4-manifold  $M$  are topologically or smoothly equivalent if there exists a homeomorphism or diffeomorphism  $g: M \rightarrow M$  such that  $s' = g \circ s$ , respectively. An *external stabilisation* of  $s: \Sigma \rightarrow M$  is given by considering the embedding of  $\Sigma$  into  $M \# (S^2 \times S^2)$ , where the connected sum is taken away from the image of  $s$ . An *internal stabilisation* of  $s$  is the embedding  $s \# U: \Sigma \# (S^1 \times S^1) \rightarrow M \# S^4$ , where  $U: S^1 \times S^1 \rightarrow S^4$  is the trivial embedding. Equivalently, it is given by adding a trivial, local tube to the embedding  $s$ .

We construct embedded surfaces that are topologically but not smoothly equivalent even after any number of external and internal stabilisations. More explicitly, we show the following.

**Theorem.** *Let  $g \in \mathbb{N}_0$  be given.*

- (1) *There exist smoothly embedded genus  $g$  orientable surfaces with normal Euler number 1 in  $\mathbb{RP}^4 \# \mathbb{CP}^2$  that are topologically but not smoothly equivalent. Moreover, they are not smoothly equivalent even after any number of internal and external stabilisations.*
- (2) *For  $n \in \mathbb{N}$  large enough, there exist smoothly embedded genus  $g$  orientable surfaces with normal Euler number 0 in  $(\mathbb{RP}^2 \times S^2) \# n(S^2 \times S^2)$  that are topologically but not smoothly equivalent. Moreover, they are not smoothly equivalent even after any number of internal and external stabilisations.*

**Remark.**

- (i) Since any pair of orientable, homeomorphic 4-manifolds are stably diffeomorphic by a result of Gompf [Gom84], the ambient manifold in our examples is necessarily non-orientable. For more details on stably exotic 4-manifolds see [KP25] for the closed case and [KOPR26] for the case with boundary. In particular one can use [KOPR26, Proposition 8.3] to show that our examples have exteriors that remain stably exotic after a connected sum with any number of copies of  $S^1 \times S^3$ , thus providing an infinite family of such pairs. On the other hand for any three topologically equivalent surfaces in a given 4-manifold at least two of them become smoothly equivalent after some number of external stabilisations as can be seen from [KOPR26, Proposition 8.1].
- (ii) The existence of stably exotic knotted surfaces in terms of isotopy, i.e. topologically isotopic but not smoothly stably isotopic surfaces, remains open [GOP26, Question 1.1]. For such examples to exist, the ambient manifold cannot be simply connected by [Gal26, Theorem 1.2] and further restrictions on the fundamental group are given by [GOP26, Theorem A].

Since an internal stabilisation increases the genus of the embedded surface by one, in both cases it suffices to construct examples of embedded spheres.

*Proof of (1).* Let  $Q$  denote Cappell–Shaneson’s exotic  $\mathbb{RP}^4$ . This is homeomorphic to  $\mathbb{RP}^4$  but not stably diffeomorphic to  $\mathbb{RP}^4$ , i.e. there does not exist a  $k$  such that  $\mathbb{RP}^4 \# k(S^2 \times S^2)$  and  $Q \# k(S^2 \times S^2)$  are diffeomorphic [CS76, p.61]. See [KPR22, Section 5.6.] for a discussion of properties of  $Q$  and further examples of exotic  $\mathbb{RP}^4$ s. There is a diffeomorphism  $f: Q \# \mathbb{CP}^2 \rightarrow \mathbb{RP}^4 \# \mathbb{CP}^2$  by [Akb84, Theorem 2], see also [BT25, Theorem 18]. Let  $s_0: S^2 \rightarrow \mathbb{RP}^4 \# \mathbb{CP}^2$  be the embedding of  $\mathbb{CP}^1$  into  $\mathbb{CP}^2$ . Similarly,

let  $s'_1: S^2 \rightarrow Q \# \mathbb{CP}^2$  be the embedding of  $\mathbb{CP}^1$  into  $\mathbb{CP}^2$  and define  $s_1 := f \circ s'_1$ . By definition the exteriors of  $s_0$  and  $s_1$  are diffeomorphic to  $\mathbb{RP}^4 \setminus \mathring{D}^4$  and  $Q \setminus \mathring{D}^4$ , respectively. In particular, since the exteriors are homeomorphic, the embeddings are topologically equivalent but not externally stably smoothly equivalent. This example of exotic embedded spheres is [BT25, Example 19].

To complete the proof of (1) it remains to show that the embeddings  $s_0$  and  $s_1$  do not become externally stably diffeomorphic after any number of internal stabilisations. Let  $s^g: \Sigma_g \rightarrow \mathbb{CP}^2$  be the internal,  $g$ -fold stabilisation of  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ . Since the embedding is characteristic, the exterior  $\mathbb{CP}_{s^g}^2$  is spin. Since  $\mathbb{CP}_{s^g}^2 \times I$  is a nullbordism of its boundary,  $\mathbb{CP}_{s^g}^2 \cup_{\partial} \mathbb{CP}_{s^g}^2 \cong \partial(\mathbb{CP}_{s^g}^2 \times I)$  has trivial signature. By Novikov additivity, it follows that gluing two copies of  $\mathbb{CP}_{s^g}^2$  along any diffeomorphism of their boundaries has trivial signature.

Let  $s_i^g$  be the internal,  $g$ -fold stabilisation of  $s_i$ . Their complements are diffeomorphic to  $\mathbb{RP}^4 \# \mathbb{CP}_{s^g}^2$  and  $Q \# \mathbb{CP}_{s^g}^2$ , respectively. In particular they admit a  $\text{pin}^+$ -structure. Any  $\text{pin}^+$ -structure will induce the unique spin-structure on  $\mathbb{CP}_{s^g}^2$  and thus a unique spin structure on the boundary. Assume there is a stably diffeomorphism  $h: \mathbb{RP}^4 \# \mathbb{CP}_{s^g}^2 \# k(S^2 \times S^2) \rightarrow Q \# \mathbb{CP}_{s^g}^2 \# k(S^2 \times S^2)$ . We pick  $\text{pin}^+$ -structures such that  $h$  is  $\text{pin}^+$ -structure preserving. In particular, it preserves the induced spin structure on the boundary. Gluing along the boundary with  $\mathbb{CP}_{s^g}^2$ , we obtain a diffeomorphism between the closed manifolds  $\mathbb{RP}^4 \# X_1 \# k(S^2 \times S^2)$  and  $Q \# X_2 \# k(S^2 \times S^2)$ , where  $X_i$  is a spin manifold consisting of two copies of  $\mathbb{CP}_{s^g}^2$  glued along the boundary. As discussed in the previous paragraph,  $X_i$  has trivial signature. Since  $X_i$  is also simply-connected, it is stably diffeomorphic to  $l$  copies of  $S^2 \times S^2$  for some  $l$ . It follows that  $\mathbb{RP}^4$  and  $Q$  are stably diffeomorphic, a contradiction. Hence  $s_0$  and  $s_1$  are internally and externally stably exotic as claimed.  $\square$

*Proof of (2).* An application of Kreck's modified surgery yields that, for  $n$  large enough, the manifolds  $X := (S^2 \times \mathbb{RP}^2) \# (n+11)(S^2 \times S^2)$  and  $X' := (S^2 \times \mathbb{RP}^2) \# K3 \# n(S^2 \times S^2)$  are diffeomorphic, see also [KP25, Theorem A]. Let  $f: X' \rightarrow X$  be such a diffeomorphism. Let  $s_0: S^2 \rightarrow (S^2 \times \mathbb{RP}^2) \# (n+11)(S^2 \times S^2)$  be given by taking  $S^2 \times \{*\}$  in  $S^2 \times \mathbb{RP}^2$ . Similarly, let  $s'_1: S^2 \rightarrow (S^2 \times \mathbb{RP}^2) \# K3 \# n(S^2 \times S^2)$  be given by taking  $S^2 \times \{*\}$  in  $S^2 \times \mathbb{RP}^2$ . We set  $s_1 := f \circ s'_1$ . Let  $M$  be the Möbius band. Then the complement of  $\nu s_0$  is diffeomorphic to  $(S^2 \times M) \# (n+11)(S^2 \times S^2)$  while the complement of  $\nu s_1$  is diffeomorphic to  $(S^2 \times M) \# K3 \# n(S^2 \times S^2)$ . Let  $s_0^g$  and  $s_1^g$  denote the  $g$ -fold internal stabilisations of  $s_0$  and  $s_1$ . The boundary of  $\nu s_i^g$  is diffeomorphic to  $S^1 \times \Sigma_g$  for  $i = 0, 1$ . Let  $W_g$  and  $W'_g$  denote the complements of  $\nu s_0^g$  and  $\nu s_1^g$ , respectively.

Assume there is a diffeomorphism taking  $s_0^g$  to  $s_1^g$  after taking a  $k$ -fold external stabilisation. Then this diffeomorphism induces a bundle isomorphism on the normal bundles and thus gives a diffeomorphism  $h$  from  $W_g \# k(S^2 \times S^2)$  to  $W'_g \# k(S^2 \times S^2)$  that respects the structure of an  $S^1$  bundle on the boundary. All bundle automorphism of the trivial  $S^1$ -bundle  $S^1 \times \Sigma_g$  extend over  $M \times \Sigma_g$  and we obtain well-defined manifolds  $X_g := W_g \cup_{\partial} (M \times \Sigma_g) \# k(S^2 \times S^2)$  and  $X'_g := W'_g \cup_{\partial} (M \times \Sigma_g) \# k(S^2 \times S^2)$ , where we choose the gluing map for  $X'_g$  by postcomposing the gluing map for  $X_g$  with  $h|_{\partial W_g}$ . Hence we also obtain a diffeomorphism  $h \cup \text{id}: X_g \rightarrow X'_g$ .

We now seek alternate descriptions of  $X_g$  and  $X'_g$  that are more amenable to the methods of modified surgery (as in [KP25]). Performing the internal stabilisations of  $S^2 \times \{*\}$  inside  $S^2 \times \mathbb{RP}^2$  yields a surface whose exterior  $C$  is a manifold with fundamental group  $\mathbb{Z}$  and boundary  $S^1 \times \Sigma_g$ . Let  $Y_g := C \cup_{\partial} M \times \Sigma_g$ . The inclusion of the boundary induces maps

$$\pi_1(S^1 \times \Sigma_g) \rightarrow \pi_1(S^1) \cong \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \cong \pi_1(C)$$

and

$$\pi_1(S^1 \times \Sigma_g) \cong \mathbb{Z} \oplus \pi_1(\Sigma_g) \xrightarrow{(\cdot 2, 1)} \mathbb{Z} \oplus \pi_1(\Sigma_g) \cong \pi_1(M \times \Sigma_g).$$

By Seifert–van Kampen it follows that  $\pi_1(Y_g)$  is isomorphic to the pushout of  $\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$  which is the fundamental group of the Klein bottle. In particular,  $\pi_1(Y_g)$  is 2-dimensional.

By construction  $X_g$  is diffeomorphic to  $Y_g \# (n+k+11)(S^2 \times S^2)$ , while  $X'_g$  is diffeomorphic to  $Y_g \# K3 \# (n+k)(S^2 \times S^2)$ . It follows again from Kreck's modified surgery theory, e.g. an application of [KP25, Theorem A, (iii)] (using that  $\pi_1(Y_g)$  is 2-dimensional), that  $X_g$  and  $X'_g$  are not stably

diffeomorphic. This contradicts the existence of  $h$ . Hence  $s_0^g$  and  $s_1^g$  are not externally stably diffeomorphic.  $\square$

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