Non-smoothable homeomorphisms of 4-manifolds

by

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Abstract

We say that a homeomorphism $f\colon X \to X'$ between two smooth manifolds is *non-smoothable* if it is not isotopic to any diffeomorphism. We produce many different examples of non-smoothable homeomorphisms of various subtleties, and discuss their properties. We show that there is a one-to-one correspondence between such non-smoothable homeomorphisms and diffeomorphic but not isotopic smooth structures on 4-manifolds, and we give an explicit construction of an infinite family of diffeomorphic but not isotopic smooth structures on the $K3$-surface.

In joint work with Roberto Ladu, we produce the first examples of non-smoothable homeomorphisms of simply-connected 4-manifolds such that the homeomorphism acts trivially on the homology of the manifold. The non-smoothability of these homeomorphisms is detected using gauge theory, and is unstable in the sense that these homeomorphisms become smoothable after sufficiently connected-summing with $S^2 \times S^2$. We use this fact to create interesting diffeomorphisms of simply-connected 4-manifolds with boundary which act trivially on the homology of the manifold, but do not arise by inserting a loop of diffeomorphisms into the collar of the boundary. This ends the joint work.

A key focus of this work is on the Casson-Sullivan invariant $\mathrm{cs}(f) \in H^3(X;\mathbb{Z}/2)$ of a homeomorphism $f\colon X \to X'$, which is the obstruction to a homeomorphism being stably pseudo-isotopic to a diffeomorphism. In particular, if a homeomorphism has non-trivial Casson-Sullivan invariant then it is non-smoothable, even after connected-summing with $S^2 \times S^2$. This invariant has a distinctly different flavour to that of gauge-theoretic invariants, and is more in-line with high-dimensional smoothing theory. Using surgery theory, we realise this invariant in a number of contexts. Firstly, we realise it for any orientable 4-manifold after a single connected-sum with $S^2 \times S^2$. Secondly, we realise it unstably for many examples of 4-manifolds, including those with finite cyclic fundamental group.

We also discuss two applications of our work on the Casson-Sullivan invariant. The first application is to embedded surfaces, and we prove that any two topologically isotopic, smoothly embedded surfaces in a simply-connected 4-manifold become smoothly isotopic after externally connected-summing with $S^2 \times S^2$ away from the surfaces.

The second application is to 3-manifolds. Let Y be a smooth 3-manifold. We consider the inclusion induced map $\widetilde{\mathrm{Diff}}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$ between the block diffeomorphism and block homeomorphism spaces, which are defined as geometric realisations of simplicial spaces and whose connected components correspond to the smooth and topological pseudo-mapping class groups, respectively. These spaces contain the classical spaces $\mathrm{Diff}(Y)$ and $\mathrm{Homeo}(Y)$ as subspaces. We show that for certain types of elliptic 3-manifolds, the inclusion induced maps $\widetilde{\mathrm{Diff}}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$ and $\mathrm{Homeo}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$ are not 1-connected.

Contents

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Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

§ 1.1 | Preamble

This thesis essentially concerns the following question, its generalisations and the applications of studying it.

Question 1.1.1. *Given a homeomorphism $f\colon X \to X'$ between smooth 4-manifolds, can f be isotoped (deformed through homeomorphisms) to a diffeomorphism?*

If the answer to Question 1.1.1 is positive, we call f *smoothable* (see Definition 2.2.1). Throughout this thesis we will see many examples of non-smoothable homeomorphisms of 4-manifolds of various flavours and subtleties. This contrasts with dimensions below four, where all homeomorphisms are smoothable. Question 1.1.1 strongly relates to the *Hauptvermutung*, which will be described in Section 1.3.

We will initially state all of the results that will be presented in this thesis before going over the background of this area e.g. what was known beforehand. However, it should be noted that we are not claiming the first constructions of non-smoothable homeomorphisms of 4-manifolds. Historically, the first examples of non-smoothable homeomorphisms of 4-manifolds are due to Friedman-Morgan [FM88]. For any notation, conventions or symbols that are not defined/explained in the text, see Section 1.4.

§ 1.2 | Results

We begin by stating the results about the existence of non-smoothable homeomorphisms before moving on to the various applications of our results.

§ 1.2.1 | Non-smoothable homeomorphisms

We start by formalising Question 1.1.1. Let X be a compact, smooth 4-manifold and denote by $\mathrm{Homeo}(X, \partial X)$ and $\mathrm{Diff}(X, \partial X)$ the topological group of homeomorphisms and diffeomorphisms of X, respectively (see Definition 2.2.2). The inclusion induced map

$$\Phi\colon \mathrm{Diff}(X, \partial X) \to \mathrm{Homeo}(X, \partial X)$$

is continuous and it induces a map

$$\Phi\colon \pi_0 \operatorname{Diff}(X, \partial X) \to \pi_0 \operatorname{Homeo}(X, \partial X)$$

on the level of mapping class groups. We say a homeomorphisms $f\colon X \to X$ is non-smoothable its mapping-class $[f]$ is not in the image of Φ, i.e. it is not isotopic to a diffeomorphism. This definition means that we require our smoothing isotopy to fix the boundary throughout. One might consider a concept of 'absolute smoothability' where we allow the isotopy to move the boundary, but this can be shown to be equivalent to the concept that we have defined here (see Proposition 4.4.2).

Previously, all known non-smoothable homeomorphisms of simply-connected 4-manifolds acted non-trivially on homology (see Section 1.3). Our first result shows that, if the boundary is non-empty, we can produce infinitely many examples of non-smoothable homeomorphisms which act trivially on homology. We denote by $\operatorname{Tor}(X, \partial X) \subset \pi_0 \operatorname{Homeo}(X, \partial X)$ the Torelli subgroup of the mapping class group, i.e. the subgroup whose induced map on homology is trivial. This theorem is joint work with Roberto Ladu [GL23].

Theorem 1.2.1. *There exists an infinite family of pairwise non-diffeomorphic, compact, oriented, smooth, simply-connected 4-manifolds $\{Z_n\}_{n\in\mathbb{N}}$ with fixed connected boundary Y such that $\operatorname{Tor}(Z_n, Y)$ is of infinite order and every non-trivial element is non-smoothable.*

The manifolds in Theorem 1.2.1 are constructed by taking a fixed manifold Z and performing knot surgeries (Definition 3.1.5) to construct the family Z_n where $Z_1 = Z$. The manifold Z is constructed as a certain codimension zero submanifold of the $K3$ surface blown-up twice (i.e. $K3\#\overline{\mathbb{CP}}^2$)).

In [GL23], Roberto Ladu and the author also constructed another family of examples which possess pairwise non-diffeomorphic boundaries, but we will not present that construction in this thesis.

Outside of the simply-connected case, there was much less known about non-smoothable homeomorphisms (see Section 1.3). In [Gal24a] the author investigated a certain invariant called the Casson-Sullivan invariant, which is an obstruction to a homeomorphism being smoothable. We briefly introduce this now. The full definition will be given in Section 5.2.

Let $f\colon X \to X'$ be a homeomorphism of smooth 4-manifolds. Then the mapping cylinder

$$W_f := \frac{X \times I \sqcup X'}{X \times I \ni (x, 1) \sim f(x) \in X'}$$

has a prescribed smooth structure on the boundary. The obstruction to extending this smooth structure to the rest of W_f is denoted by $\operatorname{cs}(f) \in H^3(X, \partial X; \mathbb{Z}/2)$, and is called the *Casson-Sullivan invariant* of f. The Casson-Sullivan invariant has the following properties.

1. A homeomorphism f is (stably) pseudo-isotopic to a diffeomorphism if and only if $\mathrm{cs}(f) = 0$. A pseudo-isotopy can be thought of as a not necessarily level-preserving isotopy (see Definition 2.1.5). By stably we mean that we allow our pseudo-isotopy to take place after taking connected-sums with $S^2 \times S^2$. This is proved in Proposition 5.4.1, Proposition 5.4.4 and Proposition 5.4.5. Proposition 5.4.5 is due to Freedman-Quinn.

2. If f and g are (stably) pseudo-isotopic, then $\mathrm{cs}(f) = \mathrm{cs}(g)$. This is proved in Proposition 5.4.1 and Proposition 5.4.4.

3. If we consider only self-homeomorphisms, then the Casson-Sullivan invariant defines a crossed homomorphism

$$\mathrm{cs}\colon \widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X) \to H^3(X, \partial X; \mathbb{Z}/2)$$

where $\widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X)$ denotes the pseudo-mapping class group, which is the group consisting of self-homeomorphisms of X under composition considered up to pseudo-isotopy relative to the boundary (see Definition 2.2.2). This is proved in Proposition 5.4.2.

The Casson-Sullivan invariant provides a tool for finding non-smoothable homeomorphisms of 4-manifolds. We now state the relevant results. First, a stable realisation result.

Theorem 1.2.2. *Let X and X' be compact, connected, smooth, orientable 4-manifolds such that $X \cong X_0\#(S^2 \times S^2)$ and $X' \cong X_0'\#(S^2 \times S^2)$ where $X_0 \approx X_0'$, and let $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$. Then there exists a homeomorphism $f\colon X \to X'$ with $\mathrm{cs}(f) = \eta$.*

In fact, we prove a stronger result which controls what these homeomorphisms look like. This will be Theorem 6.2.2.

Remark 1.2.3. Theorem 1.2.2, combined with Proposition 5.4.5 (due to Freedman-Quinn), recovers Gompf's result [Gom84] that homeomorphic, compact, connected, smooth, orientable 4-manifolds are stably diffeomorphic. The same reasoning also shows that Theorem 1.2.2 cannot be extended to the non-orientable case, as there exist compact, connected, smooth, non-orientable 4-manifolds which are homeomorphic but are not stably diffeomorphic (see [Kre84], [CS76]). If one considers only self-homeomorphisms of non-orientable smooth manifolds, then a result like Theorem 1.2.2 may still hold.

Remark 1.2.4. For such a general class of 4-manifolds, we cannot remove the stabilisation assumption in Theorem 1.2.2. For example, the Casson-Sullivan invariant is not realisable for $X = S^1 \times S^3 = X'$ (see Lemma 6.2.4).

The methods we use to prove Theorem 1.2.2 lead us the following interesting example, which demonstrates that the smoothability of a self-homeomorphism of a smooth manifold depends on the isotopy class of the smooth structure (see Section 2.3).

Corollary 1.2.5. *Let $X = (S^1 \times S^3)\#(S^1 \times S^3)\#(S^2 \times S^2)$ with the standard smooth structure and let $g\colon X \to X$ be the diffeomorphism which swaps the two $S^1 \times S^3$ connected-summands and is the identity on the $S^2 \times S^2$ connected-summand. Then there exists a smooth structure $\mathscr{S}'$ on X, which is diffeomorphic to the standard smooth structure, but is such that g is not stably pseudo-smoothable with respect to $\mathscr{S}'$. In particular, g is not smoothable with respect to $\mathscr{S}'$ (g is not isotopic to a diffeomorphism).*

We now restrict to considering self-homeomorphisms. The next result concerns a class of 4-manifolds where it is possible to remove the stabilisation assumption in Theorem 1.2.2. This theorem will be stated in terms of a certain "realisability condition", which is defined fully in Definition 8.3.1. Instead of giving an informal definition of this condition, we will state some classes of 4-manifolds for which it applies.

The Casson-Sullivan realisability condition is satisfied for 4-manifolds whose fundamental groups are in the following classes.

(i) Finite cyclic groups $\mathbb{Z}/n$.

(ii) Groups of the form $\mathbb{Z}/(2^n) \times \mathbb{Z}/2$.

(iii) Groups of the form $(\mathbb{Z}/2)^n$.

(iv) Dihedral groups D_n.

The above list is not exhaustive (for more information see Section 8.4, c.f. Proposition 9.3.2). It should also be noted that the condition holds for all manifolds with odd order fundamental groups, however, this class is not interesting as in these cases $H^3(X, \partial X; \mathbb{Z}/2) = 0$ and hence the Casson-Sullivan invariant is trivially realisable.

Theorem 1.2.6. *Let X be a compact, connected, smooth, orientable 4-manifold with $\pi_1(X)$ a good group such that X satisfies the Casson-Sullivan realisability condition (Definition 8.3.1). Then for every class $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$ there exists a homeomorphism $f\colon X \to X$ with $\mathrm{cs}(f) = \eta$.*

Remark 1.2.7. By 'good' in Theorem 1.2.6 we mean in the sense of Freedman-Quinn (see [FQ90, Chapter 2.9] or [BKK+21, §19] for a definition). It is known that the set of good groups includes elementary amenable groups, as well as groups of sub-exponential growth. In particular, all finite groups are good, so the groups listed above as satisfying the Casson-Sullivan realisability condition are also good. However, it should be noted that we do not show that the realisability condition implies that the group is good, nor vice versa.

As a corollary to the proof of this theorem we obtain the following, which shows that the homeomorphisms we realise in in Theorem 1.2.6 are homotopic to the identity. In terms of subtlety of non-smoothable homeomorphisms, this improves on Theorem 1.2.1 at the expense of only applying to non-simply-connected 4-manifolds.

Theorem 1.2.8. *Let X be a compact, connected, smooth, orientable 4-manifold with good fundamental group such that X satisfies the Casson-Sullivan realisability condition (Definition 8.3.1). Then there exists a family of homeomorphisms $\{f_\eta \colon X \to X \mid 0 \neq \eta \in H^3(X, \partial X; \mathbb{Z}/2)\}$ all distinct up to pseudo-isotopy (relative to the boundary) such that each element f_η is not stably pseudo-smoothable but each f_η is homotopic to the identity map.*

Remark 1.2.9. We quickly demonstrate that the class of manifolds for which Theorem 1.2.8 applies non-trivially is non-empty. Recall that for any finitely generated group G there exists a closed, connected, smooth, oriented 4-manifold X with $\pi_1(X) \cong G$. For $n \geq 2$ let X be such a 4-manifold for $G = \mathbb{Z}/n$ (which is in case (i) above). For n even we have that $H^3(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and hence Theorem 1.2.8 applies non-trivially to X.

§ 1.2.2 | Applications

We now discuss the various applications of the above results on non-smoothable homeomorphisms. We start with an application to stable smooth isotopy of surfaces in simply-connected 4-manifolds.

Theorem 1.2.10. *Let X be a connected, compact, simply-connected, smooth 4-manifold and let $\Sigma_1, \Sigma_2 \subset X$ be a pair of smoothly, properly embedded surfaces which are topologically isotopic relative to their boundaries. Then there exists $n \geq 0$ such that Σ_1 and Σ_2 are smoothly isotopic relative to their boundaries in $X\#(\#^n S^2 \times S^2)$, where the connected-sums are taken away from $\Sigma_1 \cup \Sigma_2$.*

Results similar to Theorem 1.2.10 have been referred to previously (see the introductions in [AKMR15, AKM+19, HKM23]), but to the best knowledge of the author, a proof of a result like this has never appeared in the literature. In the above references, it seems to be implicit that the complement of the surfaces be simply-connected, but we will need no such condition.

Remark 1.2.11. If the surface exteriors have trivial or cyclic fundamental groups, we can pair Theorem 1.2.10 with results concerning when homologous surfaces of the same genus whose exteriors have isomorphic fundamental groups are (stably) topologically isotopic (see [LW90, HK93b, Sun15b]). This gives, in some cases, that smoothly embedded homologous surfaces with the same boundaries and of the same genus and whose exteriors have isomorphic fundamental groups are stably smoothly isotopic. See Corollary 7.0.7 for the precise statement. This (in some sense) generalises the result in [AKM+19], although the result there is stronger in the sense that only one stabilisation is required.

Another application to simply-connected 4-manifolds with boundary is the following. Let X be a compact, simply-connected, smooth 4-manifold. Recall that $\mathrm{Tor}(X, \partial X)$ denotes the subgroup of the topological mapping class group consisting of homeomorphisms which act trivially on homology. It is easy to describe a class of smoothable maps in $\mathrm{Tor}(X, \partial X)$. Given a loop γ of orientation-preserving diffeomorphisms of the boundary based at the identity, we can form a diffeomorphism

$$\varphi_\gamma \colon (X, \partial X) \to (X, \partial X)$$

by inserting γ into a collar of the boundary and extending via the identity map. Such diffeomorphisms are called *generalised Dehn twists*, and, since they are supported on a collar of the boundary, they represent (smoothable) elements in $\mathrm{Tor}(X, \partial X)$. The following result shows that not all smoothable elements in $\mathrm{Tor}(X, \partial X)$ are realised by generalised Dehn twists. This theorem is joint work with Roberto Ladu [GL23].

Theorem 1.2.12. *There exists an infinite family of pairwise non-diffeomorphic compact, oriented, smooth, simply-connected* 4*-manifolds* $\{Z_n^\dagger\}_{n\in\mathbb{N}}$ *with connected boundary* Y *and* $\mathrm{Tor}(Z_n^\dagger, Y)$ *of infinite order, such that all mapping classes in* $\mathrm{Tor}(Z_n^\dagger, Y)$ *are smoothable, but only the identity map is supported on a collar of the boundary and, in particular, only the identity map is realised by a generalised Dehn twist.*

As suggested by the notation, these manifolds $Z_n^\dagger$ are closely related to the manifolds Z_n in Theorem 1.2.1. These arise as stabilisations of the manifolds Z_n. More specifically, we have

$$Z_n^\dagger \cong Z_n \#(\#^{k_n} S^2 \times S^2)$$

where $k_n \geq 0$ is an integer depending on n. Accordingly, Y is the same Y as in Theorem 1.2.1.

We now discuss an application concerning 3-manifolds. Let Y be a closed, oriented, smooth 3-manifold. Then the well known result of Cerf [Cer59, Hat83] tells us that the natural map

$$\mathrm{Diff}(Y) \xrightarrow{\simeq} \mathrm{Homeo}(Y)$$

is a homotopy equivalence. This is (part of) the reasoning behind the slogan “the categories are the same” for 3-manifolds. Closely related to the homeomorphism and diffeomorphism groups are the so-called block homeomorphism and block diffeomorphism groups $\widetilde{\mathrm{Homeo}}(Y)$ and $\widetilde{\mathrm{Diff}}(Y)$, which are defined as geometric realisations of simplicial spaces such that their connected components correspond to the topological and smooth pseudo-mapping class groups, respectively. We have the following result.

Theorem 1.2.13. *Let* Y *be a closed, elliptic* 3*-manifold such that* $H_1(Y; \mathbb{Z}/2)$ *is not trivial. Then the natural map*

$$\widetilde{\mathrm{Diff}}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$$

is not 1*-connected. In particular, it is not surjective on* π_1*.*

As a corollary, we obtain the following.

Corollary 1.2.14. *Let Y be as in Theorem 1.2.13. Then the natural map*

$$\mathrm{Homeo}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$$

is not 1-connected. In particular, it is not surjective on π_1.

It was already known that there exist reducible closed 3-manifolds Y such that the map in Corollary 1.2.14 is not 0-connected, as, for certain examples, Kwasik-Schultz [KS96], using the work of Friedman-Witt [FW86], showed that the map was not injective on π_0 (see Theorem 9.2.6). However, Theorem 1.2.13 applies to a much wider class of 3-manifolds than in [KS96], and, as we shall see in Proposition 9.2.3, this map is an isomorphism on π_0 for all manifolds in the class that Corollary 1.2.14 applies to.

The set of 3-manifolds that Theorem 1.2.13 and Corollary 1.2.14 apply to is non-empty. In particular, it contains lens spaces which have even order fundamental group. See Remark 9.0.3 for more details.

Studying this question concerning 3-manifolds leads to the following result concerning non-pseudo-smoothability of homeomorphisms of 4-manifolds. Recall that we usually consider smoothability to require the (pseudo)-isotopy to fix the boundary throughout. One can consider *absolutely (pseudo)-smoothable* homeomorphisms where we allow the (pseudo)-isotopy to move the boundary. For isotopies, this makes no difference (see Proposition 4.4.2). However, for pseudo-isotopies, these concepts are not the same.

Theorem 1.2.15. *There exists a 4-manifold X with boundary an elliptic 3-manifold such that there exists a self-homeomorphism $f\colon X \to X$ which is not pseudo-smoothable but is absolutely pseudo-smoothable.*

Finally, we have our applications to smooth structures on 4-manifolds. We say that two smooth structures are isotopic if there exists a diffeomorphism between the smooth structures which is topologically isotopic to the identity, and hence a one-parameter family of smooth structures interpolating between the two. We have similar concepts for pseudo-isotopy, and stable (pseudo)-isotopy of smooth structures (see Section 2.1 for all of these definitions). We have the following result.

Theorem 1.2.16. *There exists an infinite family of smooth structures $\{\mathscr{S}_a\}_{a\in\mathbb{Z}}$ on the $K3$ surface such that $K3_{\mathscr{S}_i} \cong K3_{\mathscr{S}_j}$ for all $i, j \in \mathbb{Z}$, but $\mathscr{S}_i$ is not isotopic to $\mathscr{S}_j$ for all $i \neq j$.*

Here the $K3$ surface is a famous 4-manifold which has many different constructions. We will present some of these in Chapter 3.

We will see in Chapter 2 that there is a one-to-one correspondence between diffeomorphic but not isotopic smooth structures on a fixed smooth manifold and isotopy classes of non-smoothable homeomorphisms (see Corollary 2.3.3, c.f. Corollary 2.3.7).

Using this correspondence, and the generalisations of this correspondence for (stable) pseudo-isotopy, we prove the following theorems, which follow from Theorem 1.2.2 and Theorem 1.2.6.

Theorem 1.2.17. *Let X be a 4-manifold such that $X \cong X'\#(S^2 \times S^2)$ for some compact, connected, smooth, orientable 4-manifold X' and let $\mathscr{S}$ denote the smooth structure on X. Then for every non-zero $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$ there exists a smooth structure $\mathscr{S}_\eta$ on X which is not stably pseudo-isotopic (relative to the boundary) to $\mathscr{S}$ but $X_{\mathscr{S}}$ is diffeomorphic to $X_{\mathscr{S}_\eta}$. Furthermore, the elements of this family of smooth structures $\{\mathscr{S}_\eta\}$ are pairwise distinct up to stable pseudo-isotopy.*

Theorem 1.2.18. *Let X be a compact, connected, smooth, orientable 4-manifold with good fundamental group which satisfies the Casson-Sullivan realisation condition. Let $\mathscr{S}$ denote the smooth structure on X. Then for every non-zero $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$ there exists a smooth structure $\mathscr{S}_\eta$ on X which is not stably pseudo-isotopic to $\mathscr{S}$ but $X_{\mathscr{S}}$ is diffeomorphic to $X_{\mathscr{S}_\eta}$. Furthermore, the elements of this family of smooth structures $\{\mathscr{S}_\eta\}$ are pairwise distinct up to stable pseudo-isotopy.*

§ 1.3 | Background

We will discuss the background in the simply-connected, closed case first, before moving onto simply-connected with boundary, and then for non-simply-connected manifolds.

§ 1.3.1 | Simply-connected, closed

Understanding the smoothability of homeomorphisms in the closed, simply-connected is simplified by us having a complete understanding of the topological mapping class group of such manifolds, due to Freedman, Perron and Quinn [Fre82, Per86, Qui86]. Their results, when combined, say that we have an isomorphism

$$\pi_0 \operatorname{Homeo}(X) \xrightarrow{\cong} \operatorname{Aut}(H_2(X), \lambda_X),$$

where $\operatorname{Aut}(H_2(X), \lambda_X)$ denotes the automorphisms of the $\mathbb{Z}$-valued intersection form λ_X. This means that understanding the smoothability of a given homeomorphism amounts to obstructing certain isometries of a smooth manifold's intersection form from being induced by a diffeomorphism. However, before continuing along this line, we explain what can be said in a positive direction regarding smoothability.

If X is a closed, simply-connected 4-manifold (or has boundary a homology sphere) with indefinite intersection form or the rank of $H_2(X)$ at most 8, then Wall [Wal64] showed that all isometries of the intersection form $\operatorname{Aut}(H_2(X\#(S^2 \times S^2)), \lambda_{X\#(S^2\times S^2)})$ can be realised by diffeomorphisms (and hence all self-homeomorphisms of $X\#(S^2 \times S^2)$ are smoothable). Ruberman and Strle [RS23] extended this result to show that any self-homeomorphism of $X\#(S^2 \times S^2)$ that acts trivially on the homology of the $S^2 \times S^2$-summand is smoothable.

Now we return to considering non-smoothability. Gauge theory, developed in the latter half of the 20th century, provides tools for obstructing certain isometries from being realised by diffeomorphisms. Using Donaldson invariants (a gauge-theoretic invariant) Friedman and Morgan constructed the first examples of non-smoothable homeomorphisms by considering self-homeomorphisms of Dolgachev surfaces [FM88], but many more since then have been found, using similar arguments. In particular, Donaldson [Don90] showed that the (standard, smooth) $K3$ surface admits a non-smoothable homeomorphism.[1] Further instances are known, see [MS97], [Bar21].

The downside of this approach is that, without an explicit construction (often related to some symmetry of the manifold), it is very hard to show that a given isometry is realisable by a diffeomorphism. This forms part of a general trend wherein gauge theory only provides obstructions, and does not help to answer questions positively.

§ 1.3.2 | Simply-connected, with boundary

For simply-connected manifolds with non-empty boundary, the topological mapping class group is more complicated than just the automorphisms of its intersection form. Work of Saeki, Orson-Powell (still crucially using the results of Perron and Quinn) gives the following isomorphism

$$\pi_0 \operatorname{Homeo}(X, \partial X) \xrightarrow{\cong} \mathcal{V}(H_2(X), \lambda_X)$$

where $\mathcal{V}(H_2(X), \lambda_X)$ denotes the group of Poincaré variations (see Definition 4.1.2), which is a more complicated algebraic object than the automorphism group of the intersection form. In particular, this means that the Torelli group, the subgroup of homeomorphisms which induce the trivial map on homology, can be non-trivial. Before the work in [GL23], however, no non-smoothable homeomorphisms had been constructed which lie in this subgroup.

We remark that Konno and Taniguchi [KT22, Theorem 1.7] have constructed non-smoothable homeomorphisms for a large class of 4-manifolds with boundary a rational homology sphere. However, these homeomorphisms do not lie in the Torelli subgroup. In fact, for connected boundary, the boundary must have $b_1(\partial X) \geq 2$, where b_1 denotes the first Betti number (see Remark 4.1.8).

We end the discussion on simply-connected 4-manifolds by noting the following. Non-smoothable homeomorphisms of simply-connected 4-manifolds are distinctively unstable by Proposition 5.4.5, due to Freedman-Quinn. The consequence of this is that there are no obstructions in the world of "formal smooth structures", in the sense there is no obstruction to smoothing the homeomorphism on the level of tangent bundles. We will discuss the concept of formal smooth structures in Chapter 5.

[1] For non-standard smooth structures on the $K3$ surface, non-smoothable homeomorphisms also exist. For example, see Theorem 1.2.16 and Corollary 2.3.3. Likely examples such as these were known to experts after [FM88].

§ 1.3.3 | Non-simply-connected case

The non-simply-connected case may well contain all of the unstable behaviour that appears in the simply-connected case, however, we shall focus on a new phenomenon that appears. Since the manifold is no longer simply-connected, an obstruction to smoothing the homeomorphism on the level of tangent bundles arises called the Casson-Sullivan invariant. We now recount what was previously known in the non-simply-connected case regarding non-smoothable homeomorphisms.

Cappell-Shaneson-Lee [CS71, Lee70] and Scharlemann-Akbulut [Sch76, Akb99] produced examples of homotopy equivalences

$$f\colon (S^1 \times S^3)\#(S^2 \times S^2) \to (S^1 \times S^3)\#(S^2 \times S^2)$$

which are not homotopic to diffeomorphisms. It was shown by Wang [Wan93, Chapter 6.2] that the Cappell-Shaneson-Lee construction could be improved to finding a non-smoothable self-homeomorphism of $(S^1 \times S^3)\#(S^2 \times S^2)$ (the reader should note that Wang states that this implies the existence of an exotic smooth structure on $(S^1 \times S^3)\#(S^2 \times S^2)$, when it actually gives a non-isotopic but diffeomorphic smooth structure; see Section 2.3 for more information). In Section 6.2 we will show that this homeomorphism has non-trivial Casson-Sullivan invariant.

History of the Casson-Sullivan invariant

The Manifold Hauptvermutung is the following conjecture: 'every homeomorphism $f\colon M \to N$ between two PL manifolds is homotopic to a PL-homeomorphism'. There is also the related conjecture, called the Isotopy Manifold Hauptvermutung: 'every homeomorphism $f\colon M \to N$ between two PL manifolds is isotopic to a PL-homeomorphism'. The work of Casson and Sullivan [Cas96, Sul96] showed that the Isotopy Manifold Hauptvermutung is true for simply-connected n-manifolds of dimension $n \geq 5$ provided that $H^3(N;\mathbb{Z}/2) = 0$, by considering a certain obstruction class $\omega \in H^3(N;\mathbb{Z}/2)$. In fact, they also showed that the (homotopy) Manifold Hauptvermutung is true for simply-connected n-manifolds of dimension $n \geq 5$ provided that $H^4(N;\mathbb{Z})$ contains no 2-torsion. Later work of Kirby-Siebenmann [KS77], crucially using the classification of PL homotopy tori by Hsiang-Shaneson and Wall [HS71, Wal70], showed that the Manifold Hauptvermutung is false in general in all dimensions $n \geq 5$, and, in particular, showed that the Casson-Sullivan class is precisely the obstruction to the Isotopy Manifold Hauptvermutung. This also gave a very slick definition of the Casson-Sullivan invariant of a homeomorphism as the Kirby-Siebenmann invariant of the mapping cylinder of the given homeomorphism.

The Casson-Sullivan invariant can similarly be defined for 4-manifolds, where, since there is no difference between the PL and smooth categories (for our purposes), we can consider it as an obstruction to smoothing homeomorphisms. Freedman-Quinn [FQ90] showed that the Casson-Sullivan invariant is the unique obstruction to stably pseudo-smoothing a homeomorphism (see Proposition 5.4.5). In the world of formal smooth structures, the Casson-Sullivan is precisely the obstruction to smoothing the homeomorphism on the level of tangent bundles.

§ 1.4 | Conventions and notation

Before we begin we will set up basic notation which will be used throughout this thesis.

(i) Let M and N be (smooth) manifolds. We write $M \cong N$ to mean M is diffeomorphic to N, $M \approx N$ to mean M is homeomorphic to N, $M \simeq N$ to mean M is a homotopy equivalent to N, and $M \simeq_s N$ to mean M is simple homotopy equivalent to N.

(ii) Let M be a topological manifold and $\mathscr{S}$ a smooth structure on M. Then we write $M_{\mathscr{S}}$ to mean the smooth manifold induced by $\mathscr{S}$ with underlying topological manifold M.

(iii) Let M be a (smooth) manifold and let Σ be a (smooth) submanifold. We write $\nu\Sigma$ for the open tubular neighbourhood of Σ in X, and write $\overline{\nu}\Sigma$ for the corresponding closed tubular neighbourhood.

(iv) Let M and N be topological manifolds of the same dimension. We write $M\#N$ to mean the connected-sum of M and N. We write $\#^k M$ for $M\# \dots_{k\text{-times}} \#M$, the k-fold connected-sum of M.

(v) We will denote the classifying space 'functor' which sends topological groups to spaces by $\mathcal{B}$ e.g. if G is a topological group, then principal G-bundles will be classified by homotopy classes of maps into the space $\mathcal{B}G$.

(vi) Our convention is to (usually) denote general manifolds as M, 3-manifolds as Y, 4-manifolds as X, and 5-manifolds as W.

(vii) Given a ring R and an R-module M, we will often denote its dual module $\mathrm{Hom}_R(M, R)$ as M^*. Given a map between two R modules $\Psi\colon M \to M'$, we will often denote the dual map as $\Psi^*\colon (M')^* \to M^*$. We consider an R-module M to be a left R-module unless stated otherwise.

§ 1.5 | Organisation

This thesis is organised as follows.

- Chapter 2: we will discuss different equivalence relations on smooth structures: diffeomorphism, isotopy, pseudo-isotopy, concordance etc. We will also explain the relation that some of these equivalence relations have with the smoothability of homeomorphisms. We will also discuss the relationship that all of these structures have with one another, with reference to what happens in other dimensions.

- Chapter 3: we will present an explicit construction of non-isotopic but diffeomorphic structures on closed, simply-connected 4-manifolds (Theorem 1.2.16) We will also discuss Fintushel and Stern's knot surgery in a wider generality, and prove that it preserves the homeomorphism types of 4-manifolds in many cases outside of the closed, simply-connected case.
- Chapter 4: we will present joint work with Roberto Ladu on non-smoothable homeomorphisms of simply-connected 4-manifolds with boundary which act trivially on homology (Theorem 1.2.1). We will also discuss how to apply this work to find diffeomorphisms of simply-connected 4-manifolds with boundary which act trivially on homology, but are not generalised Dehn twists (Theorem 1.2.12). The material from this chapter will mostly come from the paper [GL23].
- Chapter 5: we will discuss the Casson-Sullivan invariant for homeomorphisms of 4-manifolds, and prove many properties about it. We will also discuss formal smooth structures. The material from this chapter will mostly come from the paper [Gal24a].
- Chapter 6: we will prove that the Casson-Sullivan invariant can be realised stably for orientable manifolds (Theorem 1.2.2). The material from this chapter will mostly come from the paper [Gal24a] as well as the work of Ronnie Lee [Lee70] (see also [Gal24b]).
- Chapter 7: we will present an application of the stable realisation theorem from Chapter 6, that topologically isotopic surfaces in a simply-connected, smooth 4-manifold are externally, stably, smoothly isotopic (Theorem 1.2.10). The material from this chapter will mostly come from the paper [Gal24a].
- Chapter 8: we will discuss some cases where the Casson-Sullivan invariant can be realised unstably, using the surgery exact sequence (Theorem 1.2.6, Theorem 1.2.8). The material from this chapter will mostly come from the paper [Gal24a].
- Chapter 9: we will discuss an application to the unstable realisation developed in Chapter 8 to 3-manifolds (Theorem 1.2.13, Corollary 1.2.14, Theorem 1.2.15).

Chapter 2

Smooth structures on manifolds

We begin by discussing smooth structures on manifolds, and their various equivalence relations. Let us start with the basic definitions.

§ 2.1 | Equivalence relations on smooth structures on manifolds

Definition 2.1.1. Let M be a topological n-manifold. Then a *smooth atlas* $\{U_\alpha, \varphi_\alpha\}_\alpha$ for M is a collection of open subsets $\{U_\alpha \subset M\}_\alpha$ and homeomorphisms $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ such that

(i) $\cup_\alpha U_\alpha = M$,

(ii) The transition functions $\varphi_{\alpha'}^{-1} \circ \varphi_\alpha \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth for all intersections $U_\alpha \cap U_{\alpha'}$.

We say that two smooth atlases $\{U_\alpha, \varphi_\alpha\}_\alpha$ and $\{V_\beta, \psi_\beta\}_\beta$ for M are *equivalent* if the union $\{U_\alpha, \varphi_\alpha\}_\alpha \cup \{V_\beta, \psi_\beta\}_\beta$ is itself a smooth atlas for M. We define a *smooth structure* on M to be a maximal smooth atlas.

One might imagine that 'equality' would be a good equivalence relation to put on smooth structures, considering that we have already taken some closure of atlases. This turns out not to be the case, as illustrated by the following example.

Lemma 2.1.2. *There exists an uncountable number of smooth structures on the n-dimensional real space, $\mathbb{R}^n$.*

Proof. We prove this only for $n = 1$ for simplicity, but it is not hard to extend this to higher dimensions.

One can define an atlas for $\mathbb{R}$ with a single, global, chart $\varphi \colon \mathbb{R} \to \mathbb{R}$. Let $\mathscr{S}_\varphi$ denote the smooth structure which contains the atlas defined by φ, e.g. $\mathscr{S}_{\mathrm{Id}}$ denotes the standard smooth structure. Then, for $0 \le t \le 1$, let $f_t \colon \mathbb{R} \to \mathbb{R}$ be a non-smooth homeomorphism defined as

$$f_t(x) = \begin{cases} x^3 \text{ if } x \le 0 \\ x^{1+t} \text{ if } x > 0. \end{cases}$$

Then the transition function for any two of these global charts is given as

$$\varphi_{t'} \circ (\varphi_t)^{-1}(x) = \begin{cases} x \text{ if } x \leq 0 \\ x^{\frac{1+t'}{1+t}} \text{ if } x > 0 \end{cases}$$

and it is routine to see that this is not smooth (at $x = 0$) if $t \neq t'$. Hence, none of these atlases are equivalent, and so it follows that the smooth structures $\mathscr{S}_{\varphi_t}$ are all distinct. □

A consequence of the above argument is the following.

Proposition 2.1.3. *Let $n \geq 1$ and suppose that M is an n-manifold that admits a smooth structure. Then M admits uncountably many smooth structures.*

Proof. We only sketch the proof. Pick an atlas $\mathcal{A}$ on M such that $\mathcal{A}$ contains a chart $\varphi\colon U \to \mathbb{R}^n$ with the property that U contains a contractible, open region V such that V is disjoint from any other chart in $\mathcal{A}$. Now. much like in the proof of Lemma 2.1.2, create a 1-parameter family of charts φ_t for $0 \leq t \leq 1$ such that

(i) $\varphi_0 = \varphi$,

(ii) $\varphi_t = \varphi$ outside of V,

(iii) the transition function $\varphi_{t'} \circ (\varphi_t)^{-1}$ is not smooth for some point in $(\varphi_t)^{-1}(V)$ if $t \neq t'$, but is smooth otherwise.

By construction, each chart φ_t is compatible with all of the charts in $\mathcal{A}$ aside from φ. Hence, creating new atlases $\mathcal{A}_t$, where we substitute the chart φ for the chart φ_t, gives a 1-parameter family of atlases which are all not pairwise equivalent. Considering their corresponding smooth structures completes the proof. □

This leads us to consider a more coarse equivalence relation on smooth structures, namely that of diffeomorphism.

Definition 2.1.4. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be a pair of smooth structures on a topological manifold M. We denote the resulting smooth manifolds from these structures by $M_{\mathscr{S}_i}$. Then we say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are *diffeomorphic* if there exists a homeomorphism $f\colon M \to M$ such that $f^*(\mathscr{S}_2) = \mathscr{S}_1$, where $f^*(-)$ denotes the pull-back operation on smooth structures, defined in the obvious manner. We then say that f is a *diffeomorphism* between $\mathscr{S}_1$ and $\mathscr{S}_2$.

It is not hard to show that all of the smooth structures given in Proposition 2.1.3 are diffeomorphic. We leave this as an exercise. Often, diffeomorphism is the correct notion of equivalence to consider for manifolds, however it is not the only sensible one. We will now describe more equivalence relations. First, an intermediate definition.

Definition 2.1.5. Let M and N be a pair of manifolds and let $f, g\colon M \to N$ be a pair of homeomorphisms. If M has boundary then further assume that f and g restrict to a fixed homeomorphism $f_0\colon \partial M \to \partial N$. We say that f is (*relatively*) *pseudo-isotopic* to g if there exists a homeomorphism, called a (*relative*) *pseudo-isotopy*

$$F\colon M \times I \to N \times I$$

such that

(i) $F|_{\partial M \times I} = f_0 \times \mathrm{Id}$,

(ii) $F|_{M\times\{0\}} = f\colon M \times \{0\} \to N \times \{0\}$,

(iii) $F|_{M\times\{1\}} = g\colon M \times \{1\} \to N \times \{1\}$.

If we drop condition (i), then we say that f and g are *absolutely pseudo-isotopic*. We say that f and g are (*relatively/absolutely*) *isotopic* if they are (relatively/absolutely) pseudo-isotopic via a level-preserving (relative/absolute) pseudo-isotopy.

Definition 2.1.6. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be a pair of smooth structures on a topological manifold M. Then we say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are isotopic {pseudo-isotopic} if there exists a diffeomorphism $f\colon M \to M$ between $\mathscr{S}_1$ and $\mathscr{S}_2$ such that f is isotopic {pseudo-isotopic} to the identity.

One can further show that all of the smooth structures given in Proposition 2.1.3 are all isotopic (the diffeomorphism is only supported in some contractible patch).

If M is even-dimensional, then we also have stable analogues of the equivalence relations above. By stable we mean after connected-summing with $S^n \times S^n$. It turns out that this does not offer any additional equivalence relations outside of the 4-dimensional case. To explain this notion, we must make a digression to discuss connected-sums.

The construction of the connected-sum for smooth manifolds and its well-definedness are well-known (see [Kos93, §VI.1]). For topological 4-manifolds (which is all we will consider), its well-definedness was demonstrated by Freedman-Quinn [FQ90, Section 8.6] as a consequence of the "Nebenvermutung" proved in [FQ90, Theorem 8.1A] (c.f. Remark 7.0.5).

Definition 2.1.7. Let X_1, X_2 and X_1', X_2' be two pairs of connected, oriented 4-manifolds and let $f_1\colon X_1 \to X_1'$ and $f_2\colon X_2 \to X_2'$ be two orientation-preserving homeomorphisms, restricting to a fixed diffeomorphism on ∂X_1 and ∂X_2. Then, up to isotopy, we may assume by uniqueness of normal bundles [FQ90, Chapter 9.3], and the calculation of the mapping class group of S^3 [Cer68], that f_i restricts to the identity map on some disc for $i = 1, 2$. Hence we get a homeomorphism

$$f_1\#f_2\colon X_1\#X_2 \to X_1'\#X_2',$$

called the *connected-sum homeomorphism* of f_1 and f_2.

We will particularly care about the case where $X := X_1$, $X' := X'_1$, $X_2 = S^2 \times S^2 = X'_2$, $f := f_1$ and $f_2 = \mathrm{Id}_{S^2 \times S^2}$. This connected-sum operation then defines for us a map

$$f_\# := f_1 \# \mathrm{Id} \colon X\#(S^2 \times S^2) \to X'\#(S^2 \times S^2),$$

called the *stabilisation* of f. If f is not orientation-preserving (or X is not orientable) then we may still define a stabilisation by extending f onto the $S^2 \times S^2$ summand by a fixed orientation-reversing diffeomorphism rather than the identity map.

We will see a similar construction again in Section 5.4 for connected-sums over a circle.

Remark 2.1.8. Note that there are some additional subtleties concerning Definition 2.1.7 if we consider self-homeomorphisms. One would like to be able to take two self-homeomorphisms $f_1 \colon X_1 \to X_1$ and $f_2 \colon X_2 \to X_2$ and form a connected-sum self-homeomorphism $f_1 \# f_2 \colon X_1 \# X_2 \to X_1 \# X_2$, but this is, in general, not well-defined. To define a self-homeomorphism of the connected-sum, we first need to isotopy the images of the two discs (that are used to perform the connected-sum) back to their original positions. We do this by first choosing an isotopy of their centres, and then using isotopy extension [EK71], but there are many choices for such an isotopy, and different choices do not necessarily lead to isotopic self-homeomorphisms.

For certain manifolds, however, the connected-sum homeomorphism is well-defined. For example, if one of the connected-summands is $S^2 \times S^2$, as we have for stabilisations of homeomorphisms, then the connected-sum homeomorphism is well-defined. For more information, see [AKMR15].

Let us now return to smooth structures. Let $(S^2 \times S^2)_{\mathscr{T}}$ be $S^2 \times S^2$ with smooth structure $\mathscr{T}$, the standard smooth structure on $S^2 \times S^2$ given as the product of the standard smooth structures on S^2. Further let $X_{\mathscr{S}}$ be a smooth 4-manifold, and note that, up to isotopy, we may assume that $\mathscr{S}$ is standard on some topologically embedded disc $D \subset X$. Hence we may define a smooth structure (independent of choices) denoted by $\mathscr{S} \# \mathscr{T}$ on the topological connected-sum $X\#(S^2 \times S^2)$. This motivates the following definition.

Definition 2.1.9. Let $X_{\mathscr{S}_i}$ for $i = 1, 2$ be two smooth manifolds with smooth structures $\mathscr{S}_i$, respectively. Then we say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are *stably diffeomorphic* if there exists a diffeomorphism (in the sense of Definition 2.1.4)

$$f \colon X_{\mathscr{S}_1} \#(\#^k (S^2 \times S^2)_{\mathscr{T}}) \to X_{\mathscr{S}_2} \#(\#^k (S^2 \times S^2)_{\mathscr{T}})$$

for some non-negative integer k.

We say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are *stably* (pseudo-)isotopic if there exists a stable diffeomorphism (in the sense above) between $\mathscr{S}_1$ and $\mathscr{S}_2$ which is topologically (pseudo-)isotopic to the identity.

We now define a looser notion of equivalence for smooth structures.

Definition 2.1.10. Let M be a manifold. We say that smooth structures $\mathscr{S}$ and $\mathscr{S}'$ on M are *concordant* if there exists a smooth structure $\mathscr{T}$ on $M \times I$ such that $\mathscr{T}|_{M\times\{0\}} = \mathscr{S}$ and $\mathscr{T}|_{M\times\{1\}} = \mathscr{S}'$.

We say that $\mathscr{S}$ and $\mathscr{S}'$ are *sliced concordant* if they are concordant, as above, such that the projection $M \times I \to I$ is a submersion with respect to the smooth structure $\mathscr{T}$ on $M \times I$.

It is clear that a pseudo-isotopy between smooth structures gives a concordance, since the pullback of the product smooth structure by the pseudo-isotopy produces a concordance. We will only consider the stronger notion of sliced concordance in Section 5.4.1.

We will compare these various equivalence relations on smooth structures in Section 2.1.

§ 2.2 | Non-smoothable homeomorphisms

This section is devoted to introducing the concept of a non-smoothable homeomorphisms, and analogous thereof.

Definition 2.2.1. Let $f\colon M \to M'$ be a homeomorphism of smooth manifolds such that f restricts to a fixed diffeomorphism $f_0\colon \partial M \to \partial M'$. We say that f is *smoothable* if it is isotopic to a diffeomorphism. We say that f is *pseudo-smoothable* if it is pseudo-isotopic to a diffeomorphism. If M is a 4-dimensional manifold, we say that f is *stably (pseudo-)smoothable* if there exists a $k \geq 0$ such that

$$f_{\#}\colon M\#(\#^k S^2 \times S^2) \to M'\#(\#^k S^2 \times S^2)$$

is (pseudo-)smoothable.

Another perspective on this concept can be described using the following spaces, which will be important to consider in many sections throughout this thesis.

Definition 2.2.2. Let M be a (smooth) manifold. We define the *homeomorphism group of* M to be the topological group

$$\mathrm{Homeo}(M) := \{f\colon M \to M \mid f \text{ a homeomorphism}\},$$

with the compact-open topology (see [Hat02, Appendix A] c.f. [Hir76, Chapter 2]). Similarly, we define the *diffeomorphism group of* M to be the topological group

$$\mathrm{Diff}(M) := \{f\colon M \to M \mid f \text{ a diffeomorphism}\},$$

with the C^∞ topology (sometimes called the Whitney topology) For details see [Hir76, Chapter 2]).

If M has boundary then we define $\mathrm{Homeo}(M, \partial M)$ to be the sub topological group consisting of homeomorphisms which fix the boundary pointwise. Similarly, if M has an orientation then we define $\mathrm{Homeo}^+(M)$ to be the sub topological group consisting of orientation-preserving homeomorphisms. Similarly, we have the sub topological groups $\mathrm{Diff}(M, \partial M)$ and $\mathrm{Diff}^+(M)$ for the diffeomorphism group of M, although for $\mathrm{Diff}(M, \partial M)$ we also have to control the derivatives near the boundary (see [Hir76, Chapter 2]).

Taking connected components $\pi_0 \mathrm{Homeo}(M, \partial M)$ and $\pi_0 \mathrm{Diff}(M, \partial M)$ yields the *topological mapping class group* and the *smooth mapping class group*, respectively. We also define the *topological pseudo-mapping class group* $\widetilde{\pi}_0 \mathrm{Homeo}(X, \partial X)$ and the *smooth pseudo-mapping class group* by taking so-called concordance homotopy groups (see [ABK71]).

Remark 2.2.3. (i) This definition of the mapping class group means elements correspond to diffeomorphisms/homeomorphisms up to relative/absolute isotopy. The definition of the pseudo-mapping class group means that elements correspond to diffeomorphisms/homeomorphisms up to relative/absolute pseudo-isotopy.

(ii) There is a separate definition for the pseudo-mapping-class group, where we instead first define spaces $\widetilde{\mathrm{Diff}}(M, \partial M)$ and $\widetilde{\mathrm{Homeo}}(M, \partial M)$, called the *block-diffeomorphism space* and *block-homeomorphism space*, respectively. These are defined as geometric resolutions of simplicial spaces, such that there are inclusion induced maps inducing isomorphisms

$$\widetilde{\pi}_0 \mathrm{Homeo}(M, \partial M) \xrightarrow{\cong} \pi_0 \widetilde{\mathrm{Homeo}}(M, \partial M)$$

and

$$\widetilde{\pi}_0 \mathrm{Diff}(M, \partial M) \xrightarrow{\cong} \pi_0 \widetilde{\mathrm{Diff}}(M, \partial M).$$

This viewpoint will be developed further in Chapter 9 (c.f. Definition 9.1.1).

An important map is given by the following lemma, which we state without proof. See [Hir76, Chapter 2] for the details needed to produce a proof, which is clear once all of the definitions have been unravelled.

Lemma 2.2.4. *Let M be a smooth manifold. Then the inclusion induced map*

$$\Phi\colon \mathrm{Diff}(M, \partial M) \to \mathrm{Homeo}(M, \partial M)$$

is continuous.

The map Φ induces a homomorphism of the (pseudo-)mapping class groups, and this gives us an alternative formulation of smoothability. A homeomorphism $f\colon X \to X$ is smoothable if and only if $[f]$ is in the image of

$$\Phi\colon \pi_0 \mathrm{Diff}(M, \partial M) \to \pi_0 \mathrm{Homeo}(M, \partial M)$$

and pseudo-smoothable if and only if it lies in the image of the corresponding map of pseudo-mapping class groups.

In the definition of (pseudo-)smoothability it was required that the (pseudo)-isotopy fix the boundary throughout. If we relax this constraint, and allow the (pseudo-)isotopy to move the boundary, we can define a (potentially) weaker notion of smoothability.

Definition 2.2.5. Let $f\colon M \to M'$ be a homeomorphism of smooth manifolds such that f restricts to a fixed diffeomorphism $f_0\colon \partial M \to \partial M'$. We say that f is *absolutely smoothable* if it is isotopic (not relative to the boundary) to a diffeomorphism. We say that f is *absolutely pseudo-smoothable* if it is pseudo-isotopic (not relative to the boundary) to a diffeomorphism.

In the case of isotopy this distinction makes no difference, as we will see in Proposition 4.4.2. However, there exist homeomorphisms which are absolutely pseudo-smoothable but not pseudo-smoothable, which will see in Theorem 9.0.4.

§ 2.3 | Non-smoothable homeomorphisms and smooth structures

This section is devoted to writing up the correspondence between non-isotopic but diffeomorphic smooth structures and non-smoothable homeomorphisms.

§ 2.3.1 | Non-isotopic but diffeomorphic smooth structures

We start with the basic notation. In this section, like in Section 2.1, it will be convenient to let X denote a topological manifold, and to denote the smooth manifold induced by a smooth structure $\mathscr{S}$ as $X_{\mathscr{S}}$. Throughout we will assume that X already has a smooth structure on its boundary. For all of our applications, X will be a 4-manifold, and hence its boundary admits a unique smooth structure up to isotopy (see Section 2.4.2).

Let $X_{\mathscr{S}}$ be a smooth, compact 4-manifold. We denote the map induced on the mapping class groups {pseudo-mapping class groups} by the inclusion $\operatorname{Diff}(X_{\mathscr{S}}, \partial X) \hookrightarrow \operatorname{Homeo}(X, \partial X)$ as

$$\begin{aligned} \Phi &\colon \pi_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X) \to \pi_0 \operatorname{Homeo}(X, \partial X), \\ \widetilde{\Phi} &\colon \widetilde{\pi}_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X) \to \widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X). \end{aligned} \tag{2.3.1}$$

We are particularly interested in the cokernel of this map which we will write as the quotient

$$\operatorname{coker} \Phi = \frac{\pi_0 \operatorname{Homeo}(X, \partial X)}{\pi_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X)}, \quad \operatorname{coker} \widetilde{\Phi} = \frac{\widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X)}{\widetilde{\pi}_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X)}$$

which corresponds to self-homeomorphisms of X which are not topologically isotopic {pseudo-isotopic} to any self-diffeomorphism of $X_{\mathscr{S}}$. We have that this is a well-defined group, by the following lemma.

Lemma 2.3.1. *Let* Φ $\{\widetilde{\Phi}\}$ *be the map from 2.3.1. Then the subgroup* $\operatorname{im} \Phi$ $\{\operatorname{im} \widetilde{\Phi}\}$ *is a normal subgroup in* $\pi_0 \operatorname{Homeo}(X, \partial X)$ $\{\widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X)\}$.

Proof. We show that the image of $\operatorname{Diff}(X_{\mathscr{S}}, \partial X)$ in $\operatorname{Homeo}(X, \partial X)$ is normal, and the lemma then follows immediately. Let $f\colon X_{\mathscr{S}} \to X_{\mathscr{S}}$ be a diffeomorphism and let $g\colon X \to X$ be a homeomorphism. Then we want to show that $(g^{-1} \circ f \circ g)^*(\mathscr{S}) = \mathscr{S}$. It suffices to show that $f^*(g^*(\mathscr{S})) = g^*(\mathscr{S})$, i.e. that f is also a diffeomorphism with respect to the smooth structure induced by g. This is true, since f being smooth with respect to $g^*(\mathscr{S})$ is equivalent to the function $\psi \circ g \circ f \circ g^{-1} \circ \varphi^{-1}$ being a (classically) smooth map for any charts ψ, φ. By maximality of smooth structures, $\psi \circ g$ and $\varphi \circ g$ are both charts for $\mathscr{S}$, and f being a diffeomorphism with respect to $\mathscr{S}$ finishes the proof. □

Proposition 2.3.2. *Let $X_{\mathscr{S}}$ be a smooth manifold, $f\colon X \to X$ be a self-homeomorphism and let Φ $\{\widetilde{\Phi}\}$ be the map from 2.3.1. Then the smooth structures $f^*(\mathscr{S})$ and $\mathscr{S}$ are isotopic {pseudo-isotopic} if and only if $[f] \in \operatorname{im}\Phi$ $\{[f] \in \operatorname{im}\widetilde{\Phi}\}$.*

Proof. We prove this only for Φ, with the proof being exactly the same for $\widetilde{\Phi}$.

The reverse implication is straightforward. Let f be a smoothable homeomorphism relative to $\mathscr{S}$, i.e. f is topologically isotopic to a diffeomorphism $f'\colon X_{\mathscr{S}} \to X_{\mathscr{S}}$. Then the smooth structure $f^*(\mathscr{S})$ is isotopic to $(f')^*(\mathscr{S})$, and we have $(f')^*(\mathscr{S}) = \mathscr{S}$ since f' is a diffeomorphism.

Now for the forwards implication. Let f be such that $\mathscr{S}$ and $f^*(\mathscr{S})$ are isotopic. Then by the definition we have that there exists a diffeomorphism $g\colon X_{\mathscr{S}} \to X_{f^*(\mathscr{S})}$ such that g is topologically isotopic to the identity. More specifically, there exists a continuous path of homeomorphisms $g_t\colon X \to X$ such that $g_0 = \mathrm{Id}_X$ and $(g_1^{-1})^*(f^*(\mathscr{S})) = \mathscr{S}$. Consider the composition homeomorphism $f_t := f \circ g_t\colon X \to X$ and note that $f_0 = f$ and

$$f_1 = f \circ g\colon X_{\mathscr{S}} \to X_{\mathscr{S}}$$

is a diffeomorphism by construction. □

Corollary 2.3.3. *Let $X_{\mathscr{S}}$ be a smooth manifold and let $\mathcal{S}(X_{\mathscr{S}}, \partial X)$ $\{\widetilde{\mathcal{S}}(X_{\mathscr{S}}, \partial X)\}$ denote the set of isotopy {pseudo-isotopy} classes of smooth structures on X diffeomorphic to $\mathscr{S}$ restricting to the given smooth structure on ∂X. Then there is a bijection as defined below*

$$\begin{aligned} \frac{\pi_0 \operatorname{Homeo}(X, \partial X)}{\pi_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X)} &\longrightarrow \mathcal{S}(X_{\mathscr{S}}, \partial X) \\ [f] &\longmapsto f^*(\mathscr{S}), \end{aligned}$$

$$\begin{aligned} \frac{\widetilde{\pi}_0 \operatorname{Homeo}(X, \partial X)}{\widetilde{\pi}_0 \operatorname{Diff}(X_{\mathscr{S}}, \partial X)} &\longrightarrow \widetilde{\mathcal{S}}(X_{\mathscr{S}}, \partial X) \\ [f] &\longmapsto f^*(\mathscr{S}). \end{aligned}$$

Proof. The fact that the map written in the statement is well-defined and only maps the trivial element to $\mathscr{S}$ follows directly from Proposition 2.3.2. That the second condition is enough to ensure that the map is injective follows from Lemma 2.3.1, That the map is surjective follows from the definition of $\mathcal{S}(X_{\mathscr{S}}, \partial X)$ $\{\widetilde{\mathcal{S}}(X_{\mathscr{S}}, \partial X)\}$. □

Note that since the quotient is a group (by Lemma 2.3.1), this means that we get a group structure on $\mathcal{S}(X_{\mathscr{S}})$ $\{\widetilde{\mathcal{S}}(X_{\mathscr{S}})\}$ by Corollary 2.3.3.

Remark 2.3.4. There is an alternative interpretation of this section given recently by Lin-Xie [LX23], where they instead construct a space of smooth structures, and now interpret our $\mathcal{S}(X_{\mathscr{S}}, \partial X)$ as π_0 of this space. We make this more precise. We have a map

$$\mathcal{B}\mathrm{Diff}(X_{\mathscr{S}}, \partial X) \to \mathcal{B}\mathrm{Homeo}(X, \partial X)$$

induced by the inclusion and we denote the homotopy fibre of this map by $F(X_{\mathscr{S}})$. From the long exact sequence of the fibration and the fact that $\pi_i(\mathcal{B}G) = \pi_{i-1}(G)$ for any group G, we get the exact sequence

$$\pi_0 \mathrm{Diff}(X_{\mathscr{S}}, \partial X) \to \pi_0 \mathrm{Homeo}(X, \partial X) \to \pi_0(F(X_{\mathscr{S}})) \to 0.$$

Since it is clear that the first map in this sequence is Φ we have that $\pi_0(F(X_{\mathscr{S}})) = \operatorname{coker}\Phi$. We call $F(X_{\mathscr{S}})$ the *space of smooth structures on X diffeomorphic to $\mathscr{S}$*, and $\operatorname{coker}\Phi$ corresponds to its path components. Using block diffeomorphisms and block homeomorphisms (see Remark 2.2.3, Section 9.1), we can similarly define a space $\widetilde{F}(X_{\mathscr{S}})$ where $\pi_0(\widetilde{F}(X_{\mathscr{S}}))$ corresponds to $\operatorname{coker}\widetilde{\Phi}$. We will not explore this interpretation further in this thesis.

§ 2.3.2 | Stably non-isotopic but diffeomorphic smooth structures

Similarly, all of Section 2.3.1 can be considered stably. Let $(S^2 \times S^2)_{\mathscr{T}}$ be $S^2 \times S^2$ with smooth structure $\mathscr{T}$, the standard smooth structure on $S^2 \times S^2$ given as the product of the standard smooth structures on S^2. Further let $X_{\mathscr{S}}$ be a smooth 4-manifold, and note that, up to isotopy, we may assume that $\mathscr{S}$ is standard on some topologically embedded disc $D \subset X$. Hence we may define a smooth structure (independent of choices) denoted by $\mathscr{S}\#\mathscr{T}$ on the topological connected-sum $X\#S^2 \times S^2$. This motivates the following definitions.

Definition 2.3.5. Let $X_{\mathscr{S}_i}$ for $i = 1, 2$ be two smooth manifolds with smooth structures $\mathscr{S}_i$, respectively. Then a *stable diffeomorphism* from $X_{\mathscr{S}_1}$ to $X_{\mathscr{S}_2}$ is a diffeomorphism (in the sense of Definition 2.1.4)

$$f\colon X_{\mathscr{S}_1}\#(\#^k(S^2 \times S^2)_{\mathscr{T}}) \to X_{\mathscr{S}_2}\#(\#^k(S^2 \times S^2)_{\mathscr{T}})$$

for some non-negative integer k. In this case, we say that the smooth structures $\mathscr{S}_i$ are *stably isotopic* $\{$*stably pseudo-isotopic*$\}$ if $\mathscr{S}_1\#(\#^k\mathscr{T})$ is isotopic {pseudo-isotopic} to $\mathscr{S}_2\#(\#^k\mathscr{T})$.

Analogously to the unstable case, we then have the stable mapping class groups.

Definition 2.3.6. Let $\pi_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)$ $\{\widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)\}$ denote the *stable topological mapping class group of* X $\{$*stable topological pseudo-mapping class group of* $X\}$ which is defined as the quotient of $\operatorname{Homeo}(X,\partial X)$ via the equivalence relation: $f\sim g$ if f is stably isotopic {pseudo-isotopic} to g.

Continuing the analogy with the unstable case, there is then the obvious inclusion map which induces a map of the stable mapping class groups {pseudo-mapping class groups}

$$\begin{aligned}\Phi^{\mathrm{Stab}}\colon \pi_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)&\to \pi_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X),\\ \widetilde{\Phi}^{\mathrm{Stab}}\colon \widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)&\to \widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X).\end{aligned} \tag{2.3.2}$$

Again we are interested in the cokernel of this map which we write as the quotient of the stable mapping class groups {stable pseudo-mapping class groups} in the obvious manner:

$$\operatorname{coker}\Phi^{\mathrm{Stab}}=\frac{\pi_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)}{\pi_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)},\ \operatorname{coker}\widetilde{\Phi}^{\mathrm{Stab}}=\frac{\widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)}{\widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)}.$$

This cokernel corresponds to self-homeomorphisms of X, up to stable isotopy {pseudo-isotopy}, which are not stably topologically isotopic {pseudo-isotopic} to any stable self-diffeomorphism of $X_{\mathscr{S}}$. The proofs of Lemma 2.3.1, Proposition 2.3.2 and Corollary 2.3.3 follow through unchanged for Φ^{Stab}, and hence we have the following corollary.

Corollary 2.3.7. *Let $X_{\mathscr{S}}$ be a smooth manifold and let*

$$\mathcal{S}^{\mathrm{Stab}}(X_{\mathscr{S}},\partial X)\ \{\widetilde{\mathcal{S}}^{\mathrm{Stab}}(X_{\mathscr{S}},\partial X)\}$$

be the set of stable isotopy classes of smooth structures on X {stable pseudo-isotopy classes of smooth structures on X} stably diffeomorphic to $\mathscr{S}$. Then there are bijections as defined below

$$\begin{aligned}\frac{\pi_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)}{\pi_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)}&\longrightarrow \mathcal{S}^{\mathrm{Stab}}(X_{\mathscr{S}},\partial X)\\ [f]&\longmapsto f^*(\mathscr{S}).\end{aligned}$$

$$\begin{aligned}\frac{\widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Homeo}(X,\partial X)}{\widetilde{\pi}_0^{\mathrm{Stab}}\operatorname{Diff}(X_{\mathscr{S}},\partial X)}&\longrightarrow \widetilde{\mathcal{S}}^{\mathrm{Stab}}(X_{\mathscr{S}},\partial X)\\ [f]&\longmapsto f^*(\mathscr{S}).\end{aligned}$$

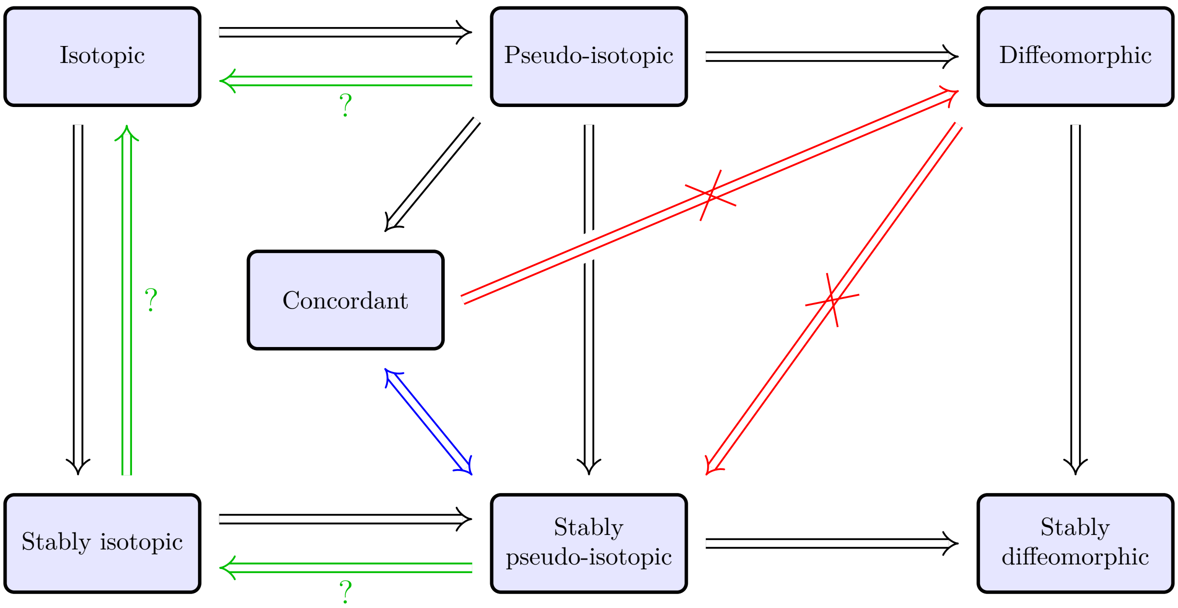
Corollary 2.3.7 immediately gives that Theorem 1.2.2 and Theorem 1.2.6 imply Theorem 1.2.17 and Theorem 1.2.18, respectively. Hence, when we prove Theorem 1.2.2 in Chapter 6 and Theorem 1.2.6 in Chapter 8, we will also prove Theorem 1.2.17 and Theorem 1.2.18.

§ 2.4 | Comparing equivalence relations on smooth structures

We finish this chapter by discussing the various equivalence relations on smooth structures, which will present in the form of charts. The trivial implications are drawn in black, whereas non-trivial implications are drawn in blue. Implications with known counter-examples are drawn with crosses in red. Unknown implications are drawn with question marks in green.

We start with the most relevant to us, the 4-dimensional case.

§ 2.4.1 | Equivalence relations on smooth structures on 4-manifolds



We will now make a series of remarks about this diagram.

- For orientable 4-manifolds, all smooth structures are stably diffeomorphic by Gompf [Gom84], and hence all smooth structures on a fixed manifold become equivalent in the bottom-right box. This does not hold in the non-orientable case by Cappell-Shaneson and Kreck [CS76, Kre84].

- The fact that concordance does not imply diffeomorphism is due to the following. By Kirby-Siebenmann (see [FQ90, Theorem 8.3B]), there exist only finitely many structures up to concordance (in fact, for simply-connected 4-manifolds all structures are concordant). Proving that concordance does not imply diffeomorphism then reduces to finding an infinite family of 4-manifolds which are all homeomorphic but pairwise not diffeomorphic (and in the simply-connected case, it suffices to construct a single exotic pair). Such examples are well-known nowadays among experts, but the first example is due to Kreck [Kre84]. The first example in the orientable case is due to Donaldson [Don87]. For infinite families outside of the simply-connected case, one can use knot surgery (see Chapter 3 for more details).

- The fact that diffeomorphism does not imply stable pseudo-isotopy is the main result of this thesis, which we achieve by realising the Casson-Sullivan invariant in many cases. It was already known that diffeomorphism does not imply pseudo-isotopy. The first examples are due to Friedman-Morgan and Donaldson [FM88, Don90], using Donaldson invariants.

- In the simply-connected case, the two horizontal unknown implications become true, due to Quinn [Qui86, GGH+23]. The vertical unknown implication then becomes false, in exactly the same cases where stable pseudo-isotopy does not imply pseudo-isotopy.

- Outside of the simply-connected case, the unknown reverse implications on the left of the diagram are all due to our current lack of understanding of the difference between topological pseudo-isotopy and topological isotopy for 4-manifolds. Budney-Gabai have produced examples of homeomorphisms pseudo-isotopic to the identity, but not isotopy to the identity, but their examples are smoothable and hence are not useful for studying whether pseudo-isotopy implies isotopy for smooth structures.

- Stable pseudo-isotopy implies concordance by the following argument. Stable pseudo-isotopy implies trivially that the smooth structures are concordant stably. Call our underlying topological manifold X. This means that we have produced a smooth structure on $X\#(\#_k(S^2\times S^2))\times I$ for some k, such that on either ends of the product it restricts to the given (stabilised) smooth structures. Since the stable tangent bundle of $S^2\times S^2$ is trivial, this means we get a formal smooth structure on $X\times I$, restricting to the given formal smooth structures on each end of the product (see Definition 5.2.5). By Kirby-Siebenmann [KS77, Theorem 10.2], this produces a smooth structure on $X\times I$, restricting to the given smooth structures on each end of the product, i.e. a concordance between the smooth structures.

- Concordance implies stable pseudo-isotopy by Proposition 5.4.5, which is due to Freedman-Quinn [FQ90, Theorem 8.2].

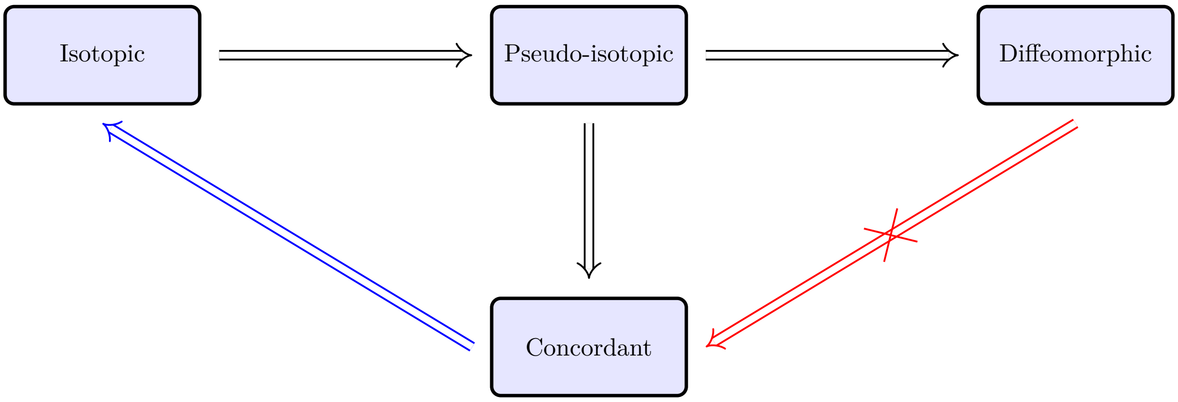
§ 2.4.2 | Equivalence relations on smooth structures in other dimensions

Dimensions three and below

In dimensions ≤ 3, the situation simplifies dramatically. Due to work of Rado, Cairns, Moise, Bing and Epstein [Rad25, Cai40, Moi52, Bin54, Eps66] every surface or 3-manifold admits a smooth structure which is unique up to isotopy. Hence, there is no need for a flowchart-like diagram in these cases[1].

[1]For 1-manifolds, the same result holds, though to the best of the author's knowledge it is not written down completely anywhere. There is an argument sketched (by the author) in unpublished notes [GNR21]. The argument is roughly to follow what Hatcher does for surfaces [Hat14].

Dimensions five and above



We make some remarks on this diagram.

- The blue implication is an immediate consquence of the celebrated "concordance implies isotopy" theorem, due to Kirby-Siebenmann [KS77, Essay I, Theorem 4.1].
- The fact that diffeomorphism does not imply concordance (and hence any of the other equivalence relations) can be seen by considering the groups of homotopy spheres (see [KM63]). For example, there are 28 different homotopy 7-spheres up to concordance, but only 15 up to diffeomorphism. We elaborate. Let Σ be a homotopy 7-sphere. Then Σ and $-\Sigma$ are diffeomorphic (the identity map is an orientation-reversing diffeomorphism between the two), but Σ is not concordant to $-\Sigma$ in general, as shown in the computations of these groups of homotopy spheres.

Chapter 3

Non-smoothable homeomorphisms of closed, simply-connected 4-manifolds

In the previous chapter, we established a correspondence between diffeomorphic but not isotopic smooth structures and non-smoothable homeomorphisms. In this chapter, we will explicitly construct diffeomorphic but not isotopic smooth structures on the $K3$ surface, without appealing to this correspondence. This has the benefit of us being able to describe the smooth structures more explicitly, although at the cost of making the construction a bit more involved. This compares with previous examples which only produce non-isotopic but diffeomorphic smooth structures via Proposition 2.3.2, i.e. non-explicitly. We now state the results of this chapter, starting with the one which was already stated in the introduction as Theorem 1.2.16.

Theorem 3.0.1. *There exists a countably-infinite family of smooth structures* $\{\mathscr{S}_a\}_{a\in\mathbb{Z}}$ *on the* $K3$ *surface such that* $K3_{\mathscr{S}_i} \cong K3_{\mathscr{S}_j}$ *for all* $i, j \in \mathbb{Z}$*, but* $\mathscr{S}_i$ *is not isotopic to* $\mathscr{S}_j$ *for all* $i \neq j$*.*

Theorem 3.0.1 is deduced using the following technical theorem, which is inspired by Fintushel-Stern and Sunukjian [FS98, Sun15a].

Theorem 3.0.2. *Let* $X_{\mathscr{S}}$ *be a closed, smooth* 4*-manifold with the following properties:*

1. *X is simply-connected.*
2. *$X_{\mathscr{S}}$ contains near-cusp embedded (see Definition 3.1.1) tori T_1 and T_2 with $[T_1] \neq [T_2]$ non-trivial in $H_2(X)$.*
3. *The tori T_k both have immersed dual spheres.*
4. *There exists a diffeomorphism $f\colon X_{\mathscr{S}} \to X_{\mathscr{S}}$ such that $f(T_1) = T_2$.*
5. *The Seiberg-Witten invariant of $X_{\mathscr{S}}$ is non-vanishing.*

Then performing knot surgery on T_1 and T_2 separately produces two smooth structures $\mathscr{S}_1, \mathscr{S}_2$ on X such that $X_{\mathscr{S}_1} \cong X_{\mathscr{S}_2}$ but $\mathscr{S}_1$ is not isotopic to $\mathscr{S}_2$.

Remark 3.0.3. Condition (1) could be weakened to account for a wider class of fundamental groups, but we will only need the simplest case for proving Theorem 3.0.1. For example, one could widen it to include $\pi_1(X) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ by using the classification results of Freedman and Quinn in [FQ90, Theorem 10.7A] or of Hambleton, Kreck and Teichner in [HKT09], respectively.

Remark 3.0.4. A useful class of manifolds to work with to ensure that condition (5) is satisfied are symplectic manifolds. All simply-connected symplectic manifolds with $b_2^+ \geq 2$ are known to have non-vanishing Seiberg-Witten invariant due to Taubes [Tau94].

§ 3.0.1 | Chapter outline

In Section 3.1 we will prove Theorem 3.0.2 using knot surgery and the Seiberg-Witten invariant. In Section 3.2, we then use this theorem to construct a family of examples on the $K3$-surface which will prove Theorem 3.0.1.

§ 3.1 | Knot surgery

We prove Theorem 3.0.2 using knot surgery and the Seiberg-Witten invariant. In Section 3.1.1 we define the knot surgery and show that under certain conditions it does not change the homeomorphism type of the initial manifold. In Section 3.1.2 we define the Seiberg-Witten invariant and state the effect of knot surgery on it. The proof of Theorem 3.0.2 is then given in Section 3.1.3.

§ 3.1.1 | The knot surgery operation

We begin with a definition.

Definition 3.1.1. Let T be an embedded torus in a smooth 4-manifold X and let α and β be embedded curves in T which represent generators of the fundamental group $\pi_1(T)$. We say that T is *near-cusp embedded* if both α and β bound discs of self-intersection number -1 in X. The curves α and β are referred to as the *vanishing circles.*

Remark 3.1.2. Equivalently, and more naturally, T being near-cusp embedded means that there exists a neighbourhood of T which looks like the neighbourhood of a cusp fibre in an elliptic fibration, where T is a regular fibre (see later in Section 3.2.1). Note that a cusp-fibre neighbourhood is simply-connected, as it results from attaching two 2-handles of framing -1 to $S^1 \times S^1 \times D^2$ which kill the generators of the fundamental group of $S^1 \times S^1$.

We now formalise two setups that we will be using throughout.

Setup 3.1.3. (Smooth setup) Let X be a compact, orientable, smooth 4-manifold (if X is not simply-connected assume $\partial X = \emptyset$), let $T \subset X$ be a near-cusp embedded torus with an immersed geometric dual sphere, and let $K \subset S^3$ be a knot.

Setup 3.1.4. (Topological setup) Let X be a compact, orientable 4-manifold (if X is not simply-connected assume $\partial X = \emptyset$), let $K \subset S^3$ be a knot, and let $T \subset X$ be an embedded torus satisfying the following two conditions.

(i) There exists a topologically immersed geometric dual sphere to T.

(ii) The inclusion induced map on fundamental groups $\pi_1(T) \to \pi_1(X)$ is trivial.

From Definition 3.1.1, it is clear that the smooth setup implies the topological setup.

Definition 3.1.5. The *knot surgery* with respect to (X, T, K) as in Setup 3.1.3 is defined to be the manifold

$$X_K := X \setminus \nu T \cup_\partial E_K \times S^1$$

where E_K is the knot exterior of K and νT denotes the (trivial) tubular neighbourhood of T in X, and the identification of the boundaries is made in the following manner. We identify the boundary of $E_K \times S^1$ with $S^1 \times S^1 \times S^1$ as follows: a meridian to K is identified with the first S^1 factor, and the 0-framed longitude to K is identified with the third S^1 factor (the leftover S^1 factor is then identified with the second S^1 factor). The boundary $\partial(X \setminus \nu T) \cong T \times S^1$ is then identified with $S^1 \times S^1 \times S^1$ in the obvious way. At this point there are still some choices to be made to determine the gluing map, but we will ignore these and instead call any resulting manifold after making these choices to be the knot surgery. For more details on this, see Fintushel and Stern [FS98, §1] or Scorpan [Sco05, §12.3].

The principal aim of knot surgery is to create exotic copies of manifolds, i.e. to find X_K homeomorphic but not diffeomorphic to X. First, we show that under certain conditions knot surgeries give homeomorphic manifolds. We begin by showing that knot surgery preserves the fundamental group.

Lemma 3.1.6. *Let X, T, K be as in Setup 3.1.4. Then $\pi_1(X_K) \cong \pi_1(X \setminus \nu T) \cong \pi_1(X)$.*

Proof. Note that the second isomorphism is actually a consequence of the first, as $X_U = X$, where U denotes the unknot in S^3. Let z denote the image inside $\pi_1(X \setminus \nu T)$ of the meridian to the torus in $\partial X \setminus \nu T \cong T \times S^1$. A straightforward Seifert-Van Kampen argument shows that $\pi_1(X_K) \cong \pi_1(X \setminus \nu T)/\langle z \rangle$. However, the existence of a geometric dual sphere for T implies that z is trivial, completing the proof. □

If the fundamental group $\pi_1(X)$ has an associated homeomorphism classification result for 4-manifolds with such a fundamental group which is sufficiently workable, we can use this to conclude that $X_k \approx X$. In this chapter, we will restrict ourselves to the cases $\pi_1(X) \cong 0, \mathbb{Z}/d$ or $\mathbb{Z} \oplus \mathbb{Z}$. We will assume that X is closed aside from in the simply-connected case.

We now briefly recap the terminology used in the relevant classification results. Let $\widetilde{X}$ denote the universal cover of X. We say that X has type (I) if $w_2(\widetilde{X}) \neq 0$, type (II) if $w_2(X) = 0$, and type (III) if $w_2(X) \neq 0$ but $w_2(\widetilde{X}) = 0$. Type (I) manifolds are called *totally non-spin*, type (II) manifolds are called *spin*, and type (III) manifolds are called *almost spin*. A *spin-structure* s is a lift of the classifying map t_X of the topological tangent bundle of X

$$\begin{array}{ccc} & & \mathcal{B}\mathrm{TOPSpin} \\ & \overset{s}{\nearrow} & \downarrow \\ X & \xrightarrow[t_X]{} & \mathcal{B}\mathrm{TOP}\,. \end{array}$$

If X is smooth then this definition matches with the (many) classical definition(s) (see [Kir89, Section IV]). In particular, a spin structure on X exists if and only if $w_2(X) = 0 \in H^2(X;\mathbb{Z}/2)$, and, if they exist, spin structures on X are in (non-canonical) bijection with $H^1(X;\mathbb{Z}/2)$. Given a spin structure s on X' and a homeomorphism $f\colon X \to X'$, we write $f^*(s)$ for the pull-back spin structure. [1]

In the simply-connected case with non-empty boundary we will need the following definition.

Definition 3.1.7. Let X_1, X_2 be oriented, compact, simply-connected 4-manifolds. Let (f, Ψ) be a pair consisting of a homeomorphism $f\colon \partial X_1 \to \partial X_2$ and an isometry of the intersection forms $\Psi\colon H_2(X_1) \to H_2(X_2)$. Then we say that f and Ψ are *compatible* if the following diagram commutes.

$$\begin{array}{ccccccc} H_2(\partial X_1) & \longrightarrow & H_2(X_1) & \longrightarrow & H_2(X_1)^* & \longrightarrow & H_1(\partial X_1) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \Psi} & & {\scriptstyle \Psi^*}\uparrow & & \downarrow{\scriptstyle f_*} \\ H_2(\partial X_2) & \longrightarrow & H_2(X_2) & \longrightarrow & H_2(X_2)^* & \longrightarrow & H_1(\partial X_2) \end{array}$$

Theorem 3.1.8 ([Fre82], [Boy86], [FQ90], [Wan93], [SW00], [HK93a], [HKT09])**.** *First fix π to be $\mathbb{Z}/d$ for some $d \geq 0$ or $\mathbb{Z} \oplus \mathbb{Z}$. Let X_1, X_2 be smooth, oriented, compact 4-manifolds with a fixed identification $\pi_1(X_1) \cong \pi \cong \pi_1(X_2)$ and assume that $\partial X_i = \emptyset$ unless π is trivial, in which case assume $\partial X_1 \cong \partial X_2$. This gives an identification $H_2(X_i;\mathbb{Z}[\pi_1(X_i)]) \cong H_2(X_i;\mathbb{Z}[\pi])$. If X_1 and X_2 have the same type, we have the following cases.*

1. *Assume $\pi \cong \mathbb{Z}/d$ for $d \neq 0$ or 1. Then given any isometry of the intersection forms*
$$\Psi\colon H_2(X_1)/\operatorname{Tors} \to H_2(X_2)/\operatorname{Tors}$$
there exists a homeomorphism $\varphi\colon X_1 \to X_2$ inducing Ψ (here Tors *denotes the torsion subgroup of the corresponding homology group).*

[1] The fact that this defines a spin structure on X is not immediately apparent (note that $f^*(s)$ is a lift of $f \circ t_{X'}$, not of t_X) but this definition will in fact make sense by arguments similar to those we will use in a later section (see Lemma 5.2.7).

2. *Assume* $\pi \cong \mathbb{Z}$ *or* $\mathbb{Z} \oplus \mathbb{Z}$. *Then given any isometry of the equivariant intersection forms*

$$\Psi \colon H_2(X_1; \mathbb{Z}[\pi]) \to H_2(X_2; \mathbb{Z}[\pi]),$$

there exists a homeomorphism $\varphi \colon X_1 \to X_2$ *inducing* Ψ.

3. *Assume* π *is trivial and, if* X_1 *is spin, let* s_{X_1}, s_{X_2} *denote the spin structures on* X_1 *and* X_2, *respectively. Then given any homeomorphism* $f \colon \partial X_1 \to \partial X_2$ *and any isometry of the intersection forms*

$$\Psi \colon H_2(X_1) \to H_2(X_2)$$

which are compatible and, if X_1 *is spin, satisfy that* $s_{X_1}|_{\partial X_1} = f^*(s_{X_2}|_{\partial X_2})$, *then there exists a homeomorphism* $\varphi \colon X_1 \to X_2$ *such that* $\varphi|_{\partial X_1} = f$ *and, if* X_1 *is spin,* φ *induces* Ψ.

We will need the following technical lemmas. After which we will prove that knot surgery preserves homeomorphism type.

Lemma 3.1.9. *Let* K *be a knot in* S^3 *and let* U *denote the unknot. Then there exists a degree one map* $\upsilon \colon E_K \to E_U$ *restricting to the identity map on the boundary.*

The above lemma is well-known. For a proof see (for example) [BBRW16, Proposition 1]. For it to make sense for υ to be the "identity map" on the boundary, we have fixed an identification of $\partial E_K = S^1 \times S^1$ using the 0-framed longitude and a meridian.

Lemma 3.1.10. *Let* X, T *and* K *be as in Setup 3.1.4. Assume* $X \setminus \nu T$ *is spin. Then* X_K *is also spin. Furthermore, given a spin structure on* $X \setminus \nu T$, *we can find a spin structure on* X_K *which matches the spin structure on* $X \setminus \nu T$.

Proof. Consider a given spin structure s on X restricted to $\partial(X \setminus \nu T) \cong S^1 \times S^1 \times S^1$. Note that s extends over $\nu T \cong T \times D^2$, and hence there are four possibilities for the restriction spin structure $s|_{S^1 \times S^1 \times S^1}$. Conversely, there are four choices of spin structures on $E_K \times S^1$, and, by restricting to the boundary, these give rise to four distinct spin structures on $\partial(E_K \times S^1) \cong S^1 \times S^1 \times S^1$. Using the degree one map $E_K \times S^1 \to E_U \times S^1 \cong T \times D^2$, we see that these are precisely the four spin structures which extend over $T \times D^2$, and so regardless of how s restricts to $\partial(X \setminus \nu T)$, we can pick a spin structure on $E_K \times S^1$ such that $s|_{X \setminus \nu T}$ extends to a spin structure on X_K.

The 'furthermore' part of the statement follows by the construction. □

Lemma 3.1.11. *Let* X, T, K *be as in Setup 3.1.4. Further assume that* $\pi_1(X) \cong \mathbb{Z}/d$ *for some* $d \geq 0$ *or* $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, *and, if* $\pi_1(X)$ *is not trivial, assume that* $\partial X = \emptyset$. *Then* X *and* X_K *have the same type.*

Proof. We first show that X_K is spin if and only if X is spin. Assume X is spin. Then $X \setminus \nu T$ is spin, and hence, by Lemma 3.1.10, X_K is spin. Assume X_K is spin. Then

$X_K \setminus \nu T = X \setminus \nu T$ is spin and again, Lemma 3.1.10, $X_U = X$ is spin. This handles types (I) and (II).

We now show that $\widetilde{X_K}$ is spin if and only if $\widetilde{X}$ is spin. Assume $\widetilde{X}$ is spin. Since $\pi_1(T) \to \pi$ given by the inclusion is trivial, T lifts into the cover as $\pi_1(X)$ many disjoint copies. Denote these copies by $T_{t\in\pi}$. Repeatedly applying Lemma 3.1.10 to all of the $T_{t\in\pi}$ shows that $(\widetilde{X})_K$ is spin, where $(\widetilde{X})_K$ denotes the manifold formed by performing the same knot surgery on all of the $T_{t\in\pi}$ in $\widetilde{X}$. Since the map $\pi_1(E_K \times S^1) \to \pi$ given by the inclusion is trivial, it follows that $\widetilde{X_K} \cong (\widetilde{X})_K$. Hence, $\widetilde{X_K}$ is spin. Assume now that $\widetilde{X_K}$ is spin. A similar reverse argument shows that $\widetilde{X_U} = \widetilde{X}$ is also spin. This handles type (III). □

Proposition 3.1.12. *Let X, T, K be as in Setup 3.1.4. Further assume that $\pi_1(X) \cong \mathbb{Z}/d$ for some $d \geq 0$ or $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, and, if $\pi_1(X)$ is not trivial, assume that $\partial X = \emptyset$. Then there exists a homeomorphism $X_K \to X$.*

In this thesis we will only ever use Proposition 3.1.12 in the simply-connected, spin case, so the reader may safely ignore the other cases if they wish.

Proof of Proposition 3.1.12. The aim here is to use Theorem 3.1.8. The first step is to establish the existence of an isometry between the (equivariant) intersection forms on X_K and X. Let $\pi := \pi_1(X)$ and note that by Lemma 3.1.6 we have an identification $\pi_1(X_K) \cong \pi$. Recall that we only have to consider the equivariant intersection form (and hence local coefficients) in the case $\pi \cong \mathbb{Z}$ or $\pi \cong \mathbb{Z} \oplus \mathbb{Z}$, but we shall write out the argument with local coefficients since this is the most complicated case.

Define F to be the identity map on $X_K \setminus \nu T$ and $\upsilon \times \mathrm{Id}$ on $E_K \times S^1$, where υ is the degree one map from Lemma 3.1.9. Note that this means that F sends the fundamental class of X_K to the fundamental class of X. Consider the following commutative diagram formed out of the Mayer-Vietoris sequences for $X_U = X$ and X_K. The horizontal maps are induced by F and its restrictions.

$$\begin{array}{ccc}
H_2(T^3;\mathbb{Z}[\pi]) & \xrightarrow{\cong} & H_2(T^3;\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow \\
H_2(X\setminus \nu T;\mathbb{Z}[\pi]) \oplus H_2(E_K\times S^1;\mathbb{Z}[\pi]) & \xrightarrow{\cong} & H_2(X\setminus \nu T;\mathbb{Z}[\pi]) \oplus H_2(E_U\times S^1;\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow \\
H_2(X_K;\mathbb{Z}[\pi]) & \xrightarrow{F_*} & H_2(X;\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow \\
H_1(T^3;\mathbb{Z}[\pi]) & \xrightarrow{\cong} & H_1(T^3;\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow \\
H_1(X\setminus \nu T;\mathbb{Z}[\pi]) \oplus H_1(E_K\times S^1;\mathbb{Z}[\pi]) & \xrightarrow{\cong} & H_1(X\setminus \nu T;\mathbb{Z}[\pi]) \oplus H_1(E_U\times S^1;\mathbb{Z}[\pi])
\end{array}$$

In the π trivial case it is easy to see that the four outer horizontal maps are isomorphisms. The fact that they are still isomorphisms with local coefficients follows because the local coefficients here are trivial in all but two cases by the assumption on T in Setup 3.1.4. In the two cases where the local coefficients are not trivial, the maps $H_*(X \setminus \nu T; \mathbb{Z}[\pi]) \to H_*(X \setminus \nu T; \mathbb{Z}[\pi])$, these maps are induced by the identity, and hence are still isomorphisms. By the five lemma, the map $F_*\colon H_2(X_K; \mathbb{Z}[\pi]) \to H_2(X; \mathbb{Z}[\pi])$ is an isomorphism. One can similarly show that F^* is an isomorphism. In the case where π is finite cyclic, F_* induces an isomorphism on the quotient by the torsion subgroups. We now show that F_* is an isometry of the equivariant intersection form.

Let λ_{X_K} and λ_X denote the equivariant intersection forms on X_K and X, respectively, and denote the fundamental classes of X and X_K as $[X]$ and $[X_K]$, respectively. Observe that, since F is degree one and F_* and F^* are isomorphisms, the following diagram commutes.

$$\begin{array}{ccc} H_2(X_K; \mathbb{Z}[\pi]) & \xrightarrow{\mathrm{PD}^{-1}} & H^2(X_K; \mathbb{Z}[\pi]) \\ \downarrow{\scriptstyle F_*} & & \uparrow{\scriptstyle F^*} \\ H_2(X; \mathbb{Z}[\pi]) & \xrightarrow{\mathrm{PD}^{-1}} & H^2(X; \mathbb{Z}[\pi]) \end{array} \tag{3.1.1}$$

To see this, note that the corresponding diagram with Poincaré duality maps rather than Poincaré duality inverses commutes by naturality of cap product and the fact that F is degree one. It is an exercise to see that changing the horizontal maps to their inverses still yields a commutative diagram. This implies that, for any $x, y \in H_2(X_K; \mathbb{Z}[\pi])$, we have

$$\begin{aligned} \lambda_X(F_*x, F_*y) =& (\mathrm{PD}^{-1} F_*x)(F_*y) \\ =& (\mathrm{PD}^{-1} F_*x)\left((\mathrm{PD}^{-1} F_*y) \frown [X]\right) \\ =& (\mathrm{PD}^{-1} F_*x)\left(F^*((\mathrm{PD}^{-1} F_*y) \frown [X_K])\right) \\ =& (F^* \mathrm{PD}^{-1} F_*x)\left(F^*((\mathrm{PD}^{-1} F_*y) \frown [X_K])\right) \\ =& \mathrm{PD}^{-1} x\left((\mathrm{PD}^{-1} y) \frown [X_K]\right) \\ =& \lambda_{X_K}(x, y). \end{aligned}$$

The first equality comes from the definition of the equivariant intersection form on X_K; the second by the definition of the equivariant Poincaré duality isomorphism; the third by the naturality of the cap product with local coefficients; the fourth by the definition of the induced map on cohomology; the fifth from the commutativity of Equation (3.1.1); and the final one by the definition of the equivariant intersection form on X. For a reference on equivariant intersection forms, cap products and Poincaré duality with local coefficients, see [Ran02, Chapter 4.5] (c.f. [Ste43]). Note that the above argument still works in the case where π is finite cyclic, as the Poincaré duality isomorphism induces an isomorphism on the quotients by the torsion subgroups.

In the closed case, it suffices to show now that X and X_K have the same type, but this was precisely established by Lemma 3.1.11. We are then done by Theorem 3.1.8.

We finish with the simply-connected case with non-empty boundary, and consider the issue of trying to extend the identity map on the boundary to a homeomorphism $X \to X_K$. First, since F_* restricts to the identity map on $X_K \setminus \nu T$, it follows that the pair $(\mathrm{Id}_{\partial X}, F_*)$ are compatible. If X is non-spin, then this means we are already done by 3. of Theorem 3.1.8, although the homeomorphism $X_K \to X$ may not induce the same isometry as given by F_* (one could likely improve this, but we will not attempt to do so). If X is spin, then Lemma 3.1.10 implies that the additional hypothesis on spin structures in 3. of Theorem 3.1.8 is satisfied, and hence we have a homeomorphism $X_K \to X$, inducing F_*. □

Remark 3.1.13. We could drop the requirement for X to be a smooth manifold in Proposition 3.1.12 by considering the Kirby-Siebenmann invariant $\mathrm{ks}(X)$ (see Definition 5.2.1), which is an obstruction to X being smoothable (i.e. it vanishes if X admits a smooth structure). The Kirby-Siebenmann invariant is additive across unions (see [FQ90, §10.2B]) and so

$$\begin{aligned}\mathrm{ks}(X_K) &= \mathrm{ks}(X \setminus \nu T) + \mathrm{ks}(E_K \times S^1)\\ &= \mathrm{ks}(X) + \mathrm{ks}(\nu T) + \mathrm{ks}(E_K \times S^1).\end{aligned}$$

Since $E_K \times S^1$ and νT both admit an obvious smooth structure, we have that

$$\mathrm{ks}(E_K \times S^1) = 0 = \mathrm{ks}(\nu T)$$

and hence $\mathrm{ks}(X_K) = \mathrm{ks}(X)$. Therefore X_K is homeomorphic to X, even if we start with a non-smoothable X. Whilst all of this true, it is not particularly useful since knot surgery is only really interesting for its effect on the smooth structure on a manifold.

§ 3.1.2 | The Seiberg-Witten invariant

As is standard, we will consider the Seiberg-Witten invariant of a manifold X as an element of the group ring $\mathbb{Z}[H_2(X)]$. The following theorem demonstrates that knot surgery can be used to produce exotic 4-manifolds, but first we recall the definition of the Seiberg-Witten invariant.

Definition 3.1.14. Let X be a smooth 4-manifold and let $\mathrm{Spin}^c(X)$ denote the set of Spin^c structures on X. The *Seiberg-Witten invariant* is defined as a map $\mathcal{SW}\colon \mathrm{Spin}^c(X) \to \mathbb{Z}$ which counts the number of solutions (the number of points in the moduli space of solutions) to the Seiberg-Witten equations on X given a particular choice of Spin^c structure. Provided that $H^2(X)$ has no 2-torsion, there is a one-to-one correspondence between $\mathrm{Spin}^c(X)$ and $\mathcal{L}_X \subset H^2(X)$ the subset consisting only of cohomology classes congruent to the second Stiefel-Whitney class $w_2(X)$ mod 2 (see Nicolaescu [Nic00, Exercise 1.3.12]). Note that for a spin manifold X this means that $\mathcal{L}_X$ consists precisely of the even classes in $H^2(X)$. In this chapter we will only ever deal with cases in which $H^2(X)$ has no 2-torsion, and so we will make this identification implicitly in what follows throughout. A cohomology class $\mathfrak{c} \in \mathcal{L}_X$ is called a *basic class* if $\mathcal{SW}(\mathfrak{c}) \neq 0$.

We can then define the *total Seiberg-Witten invariant* as an element of $\mathbb{Z}[H_2(X)]$ in the following way. Define

$$\mathcal{SW}(X) = \sum_{\mathfrak{c} \text{ basic class}} \mathcal{SW}(\mathfrak{c})\mathfrak{c}^*$$

where $\mathfrak{c}^* \in H_2(X)$ denotes the Poincaré dual of $\mathfrak{c}$.

Since we will only deal with the total invariant, we will refer to this instead as the Seiberg-Witten invariant.

We now state Fintushel and Stern's result describing how the Seiberg-Witten invariant changes after performing knot surgery.

Theorem 3.1.15 ([FS98, Theorem 1.5])**.** *Let X, T, K be as in Setup 3.1.3 and let $\varphi\colon X_K \to X$ be the homeomorphism from Proposition 3.1.12. Then*

$$\varphi_*(\mathcal{SW}(X_K)) = \mathcal{SW}(X) \cdot \Delta_K(2[T])$$

where Δ_K is the Alexander polynomial of K.

§ 3.1.3 | Proof of Theorem 3.0.2

After one final lemma, we conclude this section by providing a proof for our technical theorem.

Lemma 3.1.16. *Let X be an oriented, closed, smooth 4-manifold containing two near-cusp embedded tori T_1, T_2, each with an immersed geometric dual sphere, and let $f\colon X \to X$ be a diffeomorphism with $f(T_1) = T_2$. Fix a knot K and perform the same knot surgery with respect to K using T_1 and T_2 to form two manifolds X_1 and X_2, respectively. Then there exists a diffeomorphism of the respective knot surgeries $\widetilde{f}\colon X_1 \to X_2$ which matches f outside of a neighbourhood of T_1.*

Proof. Let νT_1 be a tubular neighbourhood of T_1 (note that this is trivial as being near-cusp embedded implies the tori T_i have zero self-intersection numbers), and we take $\nu T_2 := f(\nu T_1)$ to be the tubular neighbourhood of T_2. Choose a knot K and define the knot surgeries as before:

$$X_i = (X \setminus \nu T_i) \cup_{\theta_i} (E_K \times S^1).$$

where θ_1 is defined following the identifications made in Definition 3.1.5, and $\theta_2 := \theta_1 \circ f^{-1}$. We define θ_2 in this way to ensure that the exact same choices are made for the gluing maps in the two separate knot surgery operations (see Definition 3.1.5). Now the desired diffeomorphism is given by the map $\widetilde{f} : X_1 \to X_2$, defined as f on $X \setminus \nu T_1$ and as the identity on $E_K \times S^1$ (the map $\widetilde{f}$ is smooth and has an obvious smooth inverse). □

Proof of Theorem 3.0.2. Let X, T_1, T_2, f be as in the statement of the theorem and let K be a knot with $\Delta_K \neq 1$. Then by Theorem 3.1.15 and Proposition 3.1.12, knot surgeries with respect to K on T_1 and T_2 produce smooth manifolds X_1 and X_2 respectively, that are homeomorphic but not diffeomorphic to X. By Lemma 3.1.16 the diffeomorphism f induces a diffeomorphism $\widetilde{f}\colon X_1 \to X_2$. Let $h_1\colon X_1 \to X$ be the homeomorphism obtained via Proposition 3.1.12 and define another homeomorphism $h_2 := f \circ h_1 \circ (\widetilde{f})^{-1}$, to give the following commutative (by definition) diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h_1}\big\uparrow & & \big\uparrow{\scriptstyle h_2} \\ X_1 & \xrightarrow{\widetilde{f}} & X_2 \end{array}$$

The homeomorphisms h_1 and h_2 then induce smooth structures $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, on X, but these smooth structures are not equal as their Seiberg-Witten invariants are different. Indeed, using Theorem 3.1.15 we see that $\mathcal{SW}(X_{\mathscr{S}_1}) = \mathcal{SW}(X) \cdot \Delta_K(2[T_1])$, whereas $\mathcal{SW}(X_{\mathscr{S}_2}) = \mathcal{SW}(X) \cdot \Delta_K(2[T_2])$, and these cannot be equal, as we assumed that $\mathcal{SW}(X) \neq 0$, $[T_1] \neq [T_2]$ and $\Delta_K \neq 1$. Furthermore, any diffeomorphism

$$X_{\mathscr{S}_1} \to X_{\mathscr{S}_2}$$

isotopic to the identity must induce the trivial map on $H_2(X)$, but a diffeomorphism must send the Seiberg-Witten invariant of one to the Seiberg-Witten invariant of the other. This prevents any diffeomorphism from inducing the trivial map on $H_2(X)$ and hence prevents any diffeomorphism from being isotopic to the identity, completing the proof. □

§ 3.2 | Non-isotopic but diffeomorphic smooth structures on the K3 surface

We now use Theorem 3.0.2 to construct an infinite family of non-isotopic but diffeomorphic smooth structures on the $K3$ surface. To do this, we will need to understand the topology of the $K3$ surface more carefully. First though, we briefly recall some concepts about elliptic fibrations.

§ 3.2.1 | Elliptic fibrations

What follows here is intentionally intuitively stated and short on details. For a detailed treatment of elliptic fibrations see [GS99, §3], or for something more intermediary see [Sco05, §8].

Let X be a smooth, closed, oriented 4-manifold and C a complex curve. We say a map $\pi\colon X \to C$ is an *elliptic fibration* if each fibre $\pi^{-1}(t)$ for $t \in C$ generically looks like an elliptic curve (a 2-torus) in a holomorphic fibration. This definition allows for the existence of so-called *singular fibres* which were subsequentially classified in

[Kod63]. Non-singular fibres are referred to as *regular fibres*. For our purposes, the only singular fibres we care about are 'cusp fibres' which look like 2-spheres away from a singular point whose neighbourhood looks like the cone on a trefoil knot. We defined a cusp-fibre neighbourhood earlier in Definition 3.1.1, which gives an alternative way of viewing this. In that context, the cusp fibre results from the simultaneous collapse of both of the vanishing circles.

Two elliptic fibrations X_1, X_2 may be glued together as a *fibre-sum* by removing the neighbourhoods of two regular fibres $F_i \subset X_i$ and identifying the resulting boundaries via an orientation-reversing diffeomorphism θ. In symbols this is written as

$$X_1 \#_{\text{fib}} X_2 := (X_1 \setminus \nu F_1) \cup_\theta (X_2 \setminus \nu F_2).$$

The resulting manifold then also has the structure of an elliptic fibration. We now move on to understanding the $K3$ surface.

§ 3.2.2 | A model for the $K3$ surface

This is the standard model for the $K3$ surface where we exhibit it as an elliptic fibration explicitly.

Definition 3.2.1. We define the manifold $E(1) := \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ as the complex projective plane blown up nine times. This can be made into an elliptic fibration in the following way. First, take two generic cubics p_0 and p_1 on $\mathbb{CP}^2$ that intersect in nine points P_i for $i = 1, \ldots, 9$. For any point $Q \in \mathbb{CP}^2 \setminus \{P_1, \ldots, P_9\}$ there exists a unique $[t_0 : t_1] \in \mathbb{CP}^1$ such that Q lies on the curve $p_{[t_0:t_1]} := t_0 p_0 + t_1 p_1 = 0$. This gives the elliptic fibration map from $\mathbb{CP}^2 \setminus \{P_1, \ldots, P_9\} \to \mathbb{CP}^1$ which can be 'extended' onto the missing points by performing blow-ups at each P_i. We then form $E(n)$ by performing fibre sums on n copies of $E(1)$, and define the $K3$ surface to be $E(2)$.

The $K3$ surface is a closed, simply-connected, symplectic 4-manifold with $b_2^+ \geq 2$, with intersection form given as

$$Q_{K3} := -E_8 \oplus -E_8 \oplus H \oplus H \oplus H$$

where

$$H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & 1 & & 1 & 2 \end{bmatrix}$$

are the standard hyperbolic matrix and E_8 matrix, respectively.

To use Theorem 3.0.2 to produce an infinite family of non-isotopic but diffeomorphic smooth structures on $K3$, we need two things: a near-cusp embedded torus $T \subset K3$ with a geometric dual sphere, and a countably infinite family of diffeomorphisms $f_i\colon K3 \to K3$ such that $(f_i)_*[T] \neq (f_j)_*[T]$ for all $i \neq j$. First, we describe the torus.

Setup 3.2.2. For this setup, we follow Gompf and Mrowka [GM93, §2]. We start by taking $E(1)$ such that its only singular fibres are cusp fibres (see [GS99, Corollary 7.3.22]) and as in Definition 3.2.1 we remove neighbourhoods of regular fibres and identify the boundaries together, but this time we will be more explicit. Choose a copy of the regular fibre F near a cusp fibre and identify its neighbourhood with $F \times D^2$, then identify $F = S^1 \times S^1$ such that the vanishing circles specifically correspond to $S^1 \times \{1\}$ and $\{1\} \times S^1$. After removing the regular fibre neighbourhoods, we are left with two manifolds which we call L and L'. Write $\partial L = F \times S^1 = S^1 \times S^1 \times S^1$, identifying in the obvious manner, and glue this to $\partial L'$ via

$$\partial L = S^1 \times S^1 \times S^1 \ni (z_1, z_2, z_3) \to (z_1, z_2, z_3^*) \in S^1 \times S^1 \times S^1 = \partial L'.$$

Now we define the torus T to be the regular fibre $F \times \{1\} \subset \partial L$.

Since this will be useful in the following proposition, we briefly recall the mapping class group of the 3-torus T^3. Since T^3 is an Eilenberg-Maclane space, the group of self-homotopy equivalences is isomorphic to the group of automorphisms of $\mathbb{Z}^3$, which is isomorphic to $GL(3, \mathbb{Z})$. Since T^3 is orientable, sufficiently large and irreducible (i.e. Haken) the main theorem in [Hat76] by Hatcher implies that the mapping class group is given as $\pi_0(\mathrm{Diff}^+(T^3)) \cong \mathrm{SL}(3, \mathbb{Z})$. We now describe how a matrix $M \in \mathrm{SL}(3, \mathbb{Z})$ determines a self-diffeomorphism of T^3 in this identification. Choose a basis for $H_1(T^3)$, which then gives a basis for the universal cover $\mathbb{R}^3$ considered as a vector space over $\mathbb{R}$. The matrix M then describes a self-diffeomorphism of $\mathbb{R}^3$ which descends to a diffeomorphism on T^3.

Proposition 3.2.3. *There exists an infinite family of diffeomorphisms $f_i\colon K3 \to K3$ such that $(f_i)_*[T] \neq (f_j)_*[T]$ for all $i \neq j$.*

Proof. First we construct an infinite family of diffeomorphisms of the 3-torus, which we will then extend to a family of diffeomorphisms of $K3$. The mapping class group of the 3-torus is $SL(3, \mathbb{Z})$ as described above. Choose the basis

$$\{e_1 := S^1 \times \{1\} \times \{1\}, e_2 := \{1\} \times S^1 \times \{1\}, e_3 := \{1\} \times \{1\} \times S^1\}$$

for $H_1(T^3)$ so that we can write isotopy classes of orientation-preserving diffeomorphisms of T^3 via matrices in $SL(3, \mathbb{Z})$. For any $a \in \mathbb{Z}$ let

$$M_a := \begin{bmatrix} 1 & 1 & 0 \\ a & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \in SL(3, \mathbb{Z}).$$

By a result of Matumoto [Mat84, Theorem 1], all orientation-preserving diffeomorphisms of $\partial(E(1) \setminus \nu F)$ extend over $E(1) \setminus \nu F$ and so the diffeomorphism represented by M_a extends to a diffeomorphism $f_a \colon K3 \to K3$. The induced map from the inclusion $H_2(T^3) \to H_2(L)$ is injective (the three generators map to generators in the hyperbolic summands of the intersection form [GS99, §3.1]) and so it suffices to show that the images of T under the family M_a are all homologically distinct in $H_2(T^3)$. We pick a basis for $H_2(T^3)$ by taking wedge products of our basis for $H_1(T^3)$, i.e. $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$, with $[T] = e_1 \wedge e_2$. A simple computation gives

$$M_a(e_1 \wedge e_2) = \begin{pmatrix} 1-a \\ -1 \\ a-2 \end{pmatrix} \in H_2(T^3) \cong \mathbb{Z}^3$$

and so the images are homologically distinct, completing the proof. □

Lemma 3.2.4. *The regular fibre T in $K3$ has an immersed dual sphere.*[2]

Proof. Consider one of the self-intersection number -1 spheres in $E(1)$ corresponding to one of the blow-ups. This sphere intersects every regular fibre of $E(1)$ exactly once. When we form $E(2) = L \cup L'$ there is now a new sphere formed from the sum of two of these spheres (one from each copy of $E(1)$) which now has self-intersection number -2 but still intersects the regular fibre geometrically once. □

§ 3.2.3 | Proof of Theorem 3.0.1

The proof of the existence of non-isotopic but diffeomorphic smooth structures on the $K3$ surface using Theorem 3.0.2 now follows easily.

Proof of Theorem 3.0.1. Let T again be as in Setup 3.2.2 and let $\{f_a\}_{a\in\mathbb{Z}}$ be the family of diffeomorphisms obtained from Proposition 3.2.3. Pick a knot K with $\Delta_K \neq 1$. Then performing knot surgery using K separately on each tori in $\{f_a(T)\}_{a\in\mathbb{Z}}$ produces a countably-infinite family of smooth structures $\{\mathscr{S}_a\}_{a\in\mathbb{Z}}$ on $K3$. To complete the proof, we need to verify all of the properties required to use Theorem 3.0.2 on all of the pairs of tori in $\{f_a(T)\}_{a\in\mathbb{Z}}$.

Property (1) is clear. Property (2) follows from observing that since T is already near-cusp embedded, all of the images $f_a(T)$ must also be since the diffeomorphism sends a near-cusp neighbourhood to another near-cusp neighbourhood. By the same argument, property (3) follows directly from Lemma 3.2.4, and property (4) follows from Proposition 3.2.3 using the diffeomorphism $f_b \circ {f_a}^{-1}$ for $a, b \in \mathbb{Z}$, which clearly sends $f_a(T)$ to $f_b(T)$. Finally, property (5) holds since $K3$ is symplectic, and hence holds as $b_2^+ \geq 2$ (see Remark 3.0.4). In fact $\mathcal{SW}(K3) = 1$, also due to Taubes [Tau94].

Since all of the properties hold for all pairs of tori in $\{f_a(T)\}_{a\in\mathbb{Z}}$, applying Theorem 3.0.2 to the pair $\{f_a(T), f_b(T)\}$ for $a, b \in \mathbb{Z}$ using the knot K produces smooth

[2] Daniel Ruberman tells me that, in fact, the regular fibre can be arranged to have an embedded dual sphere, but we will not need this.

structures $\mathscr{S}_a$ and $\mathscr{S}_b$ from the family above which now must be diffeomorphic but not isotopic. Repeating this for all such pairs of tori completes the proof that all of the smooth structures in $\{\mathscr{S}_a\}_{a\in\mathbb{Z}}$ are pairwise diffeomorphic but pairwise not isotopic. $\square$

Chapter 4

Non-smoothable homeomorphisms of simply-connected 4-manifolds with boundary

This chapter concerns non-smoothable homeomorphisms of simply-connected 4-manifolds which act trivially on the homology of the manifold. In particular, we will prove Theorem 1.2.1 and Theorem 1.2.12 from the introduction. The work in this chapter is joint with Roberto Ladu, and all of the content can be found, sometimes verbatim, from [GL23].

Recall that the Torelli subgroup $\mathrm{Tor}(X, \partial X) \subset \pi_0 \,\mathrm{Homeo}^+(X, \partial X)$ is the subgroup of the topological mapping class group consisting of homeomorphisms whose induced map on homology is trivial. Recall that, by the work of Perron-Quinn [Per86, Qui86] the Torelli subgroup in the closed, simply-connected case is always trivial. We wish to study non-smoothable homeomorphisms which lie in the Torelli subgroup, and hence we must consider 4-manifolds with non-empty boundary.

We start by restating the theorems that we will prove. First, Theorem 1.2.1.

Theorem 4.0.1. *There exists an infinite family of pairwise non-diffeomorphic, compact, oriented, smooth, simply-connected* 4*-manifolds* $\{Z_n\}_{n\in\mathbb{N}}$ *with fixed connected boundary* Y *such that* $\mathrm{Tor}(Z_n, Y)$ *is infinite order and every non-trivial element is non-smoothable.*

In [GL23] we produced two such families, one such that the boundaries are pairwise not diffeomorphic, and another family $\{Z_n\}_{n\in\mathbb{N}}$ such that the boundaries ∂Z_n are all diffeomorphic and the Z_n are all homeomorphic relative to their boundaries (Theorem 4.3.1). Furthermore, the first of these families is minimal in the sense that the produced manifolds have the simplest possible intersection forms. See Remark 4.1.8 for more details. We will only present the second of these families in this chapter. For more details on the first family, see [GL23].

Recall that the non-smoothability of a homeomorphisms for a manifold with boundary means that it is not isotopic (relative to the boundary) to any diffeomorphism. As stated previously, this actually implies that the homeomorphism is non-smoothable in a stronger sense, in that it is also not isotopic to any diffeomorphism even through isotopies that do not fix the boundary. We will show this fact in Section 4.4 (see Definition 2.2.1 and Definition 2.2.5 for the relevant definitions).

Recall that generalised Dehn twists are one way of producing smoothable elements of $\mathrm{Tor}(X, \partial X)$. These are constructed by inserting a loop of diffeomorphisms of the boundary into a collar of the boundary of the 4-manifold and extending by the identity. We restate Theorem 1.2.12, which says that this construction does not provide all smoothable elements of the Torelli group.

Theorem 4.0.2. *There exists an infinite family of pairwise non-diffeomorphic compact, oriented, smooth, simply-connected 4-manifolds $\{Z_n^\dagger\}_{n\in\mathbb{N}}$ with connected boundary Y and $\mathrm{Tor}(Z_n^\dagger, Y)$ of infinite order, such that all mapping classes in $\mathrm{Tor}(Z_n^\dagger, Y)$ are smoothable, but only the identity map is supported on a collar of the boundary and, in particular, only the identity map is realised by a generalised Dehn twist.*

§ 4.0.1 | Chapter outline

In Section 4.1 we recall the classification of $\mathrm{Tor}(X, \partial X)$ in terms of algebraic objects called variations, and prove a key lemma (Lemma 4.1.9) which we will use to detect elements of the Torelli group. In Section 4.2 we prove technical conditions under which we can guarantee the existence of non-smoothable elements of the Torelli group. In Section 4.3 we use the conditions from the previous section to produce our family of examples and hence prove Theorem 4.0.1. Then in Section 4.4 we consider generalised Dehn twists and prove Theorem 4.0.2.

§ 4.0.2 | Specific chapter acknowledgements

I thank Roberto Ladu for agreeing to me including this joint work in my thesis. I specifically thank Mark Powell, Simona Veselá and Burak Özbağcı for their suggestions for the paper [GL23], where this material comes from.

§ 4.1 | Variations

§ 4.1.1 | Definitions

The aim of this section is to describe the classification of homeomorphisms up to isotopy for simply-connected, topological, oriented 4-manifolds with boundary. This classification is due to the work of Osamu Saeki, Patrick Orson, and Mark Powell [Sae06, OP23]. We will begin by defining what a variation is, which is the central object involved in the classification.

Definition 4.1.1. Let X be a simply-connected, oriented 4-manifold with boundary and let $f \in \mathrm{Homeo}^+(X, \partial X)$ be an orientation-preserving homeomorphism relative to the boundary ∂X. Then the *variation induced by* f, denoted as Δ_f, is defined as

$$\begin{aligned}\Delta_f \colon H_2(X, \partial X) &\to H_2(X)\\ [\Sigma] &\mapsto [\Sigma - f(\Sigma)],\end{aligned}$$

where Σ denotes a relative 2-chain. Note that the homology class $[\Sigma - f(\Sigma)]$ does not depend on the choice of representative relative 2-chain Σ [OP23, Section 2.2].

We can also define variations without reference to a homeomorphism.

Definition 4.1.2. Let X be a simply-connected, oriented 4-manifold with boundary and let $\Delta \colon H_2(X, \partial X) \to H_2(X)$ be a homomorphism. Then we say that Δ is a *Poincaré variation* if

$$\Delta + \Delta^! = \Delta \circ j_* \circ \Delta^! \colon H_2(X, \partial X) \to H_2(X),$$

where j is the inclusion map of pairs $(X, \emptyset) \to (X, \partial X)$ and $\Delta^!$ denotes the 'umkehr' homomorphism to Δ, defined as the following composition:

$$\Delta^! \colon H_2(X, \partial X) \xrightarrow{\mathrm{PD}^{-1}} H^2(X) \xrightarrow{\mathrm{ev}} H_2(X)^* \xrightarrow{\Delta^*} H_2(X, \partial X)^* \xrightarrow{\mathrm{ev}^{-1}} H^2(X, \partial X) \xrightarrow{\mathrm{PD}} H_2(X).$$

Following the notation of Orson-Powell, we will denote the set of Poincaré variations of $(X, \partial X)$ as $\mathcal{V}(H_2(X), \lambda_X)$, where λ_X denotes the intersection form of X. This notation is used because it is shown in [OP23, Section 7] that the set of variations only depends on the isometry class of the intersection form $(H_2(X), \lambda_X)$, rather than on the 4-manifold specifically.

We can give $\mathcal{V}(H_2(X), \lambda_X)$ the structure of a group due to the following lemma of Saeki.

Lemma 4.1.3 ([Sae06, Lemma 3.5])**.** *The set $\mathcal{V}(H_2(X), \lambda_X)$ forms a group with multiplication given by*

$$\Delta_1 \cdot \Delta_2 := \Delta_1 + (\mathrm{Id} - \Delta_1 \circ j_*) \circ \Delta_2,$$

identity the zero homomorphism, and inverse given by

$$\Delta^{-1} = -(\mathrm{Id} - \Delta \circ j_*) \circ \Delta.$$

Further, we have

Lemma 4.1.4 ([Sae06, Lemma 3.2])**.** *Let X be a compact, simply-connected, oriented, topological 4-manifold with boundary ∂X and let $f \in \mathrm{Homeo}^+(X, \partial X)$. Then Δ_f is a Poincaré variation.*

The converse of the above result, that all Poincaré variations are induced via homeomorphisms, is given by [OP23, Theorem A].

§ 4.1.2 | The Torelli group

The map which sends a homeomorphism to its variation gives a factorisation of the map which takes the induced automorphism of the form for a homeomorphism:

$$\pi_0 \operatorname{Homeo}^+(X, \partial X) \xrightarrow{f \mapsto \Delta_f} \mathcal{V}(H_2(X), \lambda_X) \xrightarrow{\Delta \mapsto \operatorname{Id} - \Delta \circ q} \operatorname{Aut}(H_2(X), \lambda_X), \tag{4.1.1}$$

where $q \colon H_2(X) \to H_2(X, \partial X)$ is the quotient map. It is the result of Freedman–Perron–Quinn [Fre82, Per86, Qui86] that, for a *closed*, simply-connected 4-manifold X, the above composition is a bijection. In particular, all homeomorphisms of a closed simply-connected 4-manifold that map to the trivial element of $\operatorname{Aut}(H_2(X), \lambda_X)$ are isotopic to the identity map. For manifolds with non-empty boundary, the classification is more subtle [OP23, Theorem A], and in particular we can have homeomorphisms that are not isotopic to the identity but still induce the trivial element of $\operatorname{Aut}(H_2(X), \lambda_X)$ under 4.1.1.

Definition 4.1.5. Let X be a compact, simply-connected, oriented, 4-manifold with boundary ∂X. We define the *Torelli group* $\operatorname{Tor}(X, \partial X) \subset \pi_0 \operatorname{Homeo}^+(X, \partial X)$ to be the subgroup of homeomorphisms that induce the trivial element of $\operatorname{Aut}(H_2(X), \lambda_X)$ under 4.1.1.

The subgroup of variations which are induced by elements in the Torelli group is exactly the subgroup of variations satisfying that $\Delta \circ q \colon H_2(X) \to H_2(X)$ is the zero map (note that a variation being induced by a Torelli group element also implies that $q \circ \Delta$ is also the zero map). For such Poincaré variations, we can construct a skew-symmetric pairing in the following way. Let Δ be a Poincaré variation. Then this gives rise to a map

$$(\eta_\Delta)^{\mathrm{ad}} \colon H_1(\partial X) \to H_2(\partial X) \cong H_1(\partial X)^*$$

(note that the last isomorphism is given by Poincaré duality and universal coefficients) by first lifting an element in $H_1(\partial X)$ to an element in $H_2(X, \partial X)$, mapping to $H_2(X)$ using Δ (note that this image does not depend on the choice of lift) and then noting that, since $q \circ \Delta = 0$ and $H_3(X, \partial X) = 0$, this element lifts uniquely to an element in $H_2(\partial X)$. As suggested by the notation, we can interpret this map as the adjoint of a pairing:

$$\eta_\Delta \colon H_1(\partial X) \times H_1(\partial X) \to \mathbb{Z}$$

and it is stated in [Sae06, Proposition 4.2] that this form is skew-symmetric. To see this, it is enough to verify that $\eta_\Delta^{\mathrm{ad}}(x)(x) = 0$ for any $x \in H_1(\partial X)$, and this fact is geometrically clear from the definition of η^{ad}. More crucially, we can go the other way. Let $\eta \colon H_1(\partial X) \times H_1(\partial X) \to \mathbb{Z}$ be a skew-symmetric pairing. Then we can define a variation Δ_η as the following composition:

$$H_2(X, \partial X) \xrightarrow{\partial} H_1(\partial X) \xrightarrow{\eta^{\mathrm{ad}}} H_1(\partial X)^* \xrightarrow{\mathrm{ev}^{-1}} H^1(X) \xrightarrow{PD} H_2(\partial X) \xrightarrow{i_*} H_2(X) \tag{4.1.2}$$

where the map ∂ denotes the connecting homomorphism in the long exact sequence of the pair and $i\colon \partial X \to X$ denotes the inclusion. We have the following sequence, due to Saeki.

Proposition 4.1.6 ([Sae06, Proposition 4.2],[OP23, Theorem 7.13]). *Let X be a compact, simply-connected, oriented, topological 4-manifold with* connected *boundary ∂X. Then the following is a short exact sequence:*

$$0 \to \wedge^2 H_1(\partial X)^* \to \mathcal{V}(H_2(X), \lambda_X) \to \mathrm{Aut}(H_2(X), \lambda_X) \to 0.$$

So it follows that, in the connected boundary case, we have that the variations which induce the trivial map on homology are in one-to-one correspondence with skew-symmetric forms on $H_1(\partial X)$.

We have the following classification of the Torelli group, due to Orson-Powell.

Theorem 4.1.7 ([OP23, Corollary D]). *Let X be a compact, simply-connected, oriented 4-manifold with connected boundary ∂X. Then there is an isomorphism of groups:*

$$\mathrm{Tor}(X, \partial X) \xrightarrow{\cong} \wedge^2 H_1(\partial X)^*,$$
$$[f] \mapsto \eta_{\Delta_f}.$$

Remark 4.1.8. It follows from this that $\mathrm{Tor}(X, \partial X)$ is non-trivial if and only if $b_1(\partial X) \geq 2$. In fact, we can say more. Since X is simply-connected, it must also have $b_2(X) \geq b_1(\partial X)$ and, if $b_2(X) = b_1(\partial X)$, vanishing intersection form. This follows from the exact sequence

$$0 \to H_2(\partial X) \to H_2(X) \xrightarrow{\lambda^{\mathrm{ad}}} H_2(X)^* \to H_1(\partial X) \to 0,$$

where λ^{ad} is the adjoint of the intersection form and the penultimate map is the composition of the inverse of the evaluation map, Poincaré duality and the connecting morphism of the long exact sequence of the pair. After tensoring with $\mathbb{Q}$ and using that $b_2(X) = b_1(\partial X)$, the claim is clear. It follows that examples of 4-manifolds with $\mathrm{Tor}(X, \partial X)$ non-trivial must have $b_2(X) \geq 2$.

§ 4.1.3 | Applying variations to closed manifolds

Let W be a simply-connected, oriented manifold with boundary ∂W. In Section 4.2 we will want to use variations to prove that a homeomorphism $f\colon (W, \partial W) \to (W, \partial W)$ is non-smoothable. In doing so, we will need the following lemma.

Lemma 4.1.9. *Let W_1 be a simply-connected, oriented 4-manifold with boundary $\partial W_1 \cong Y$, W_2 be an oriented 4-manifold with boundary $\partial W_2 \cong -Y$ and let $X := W_1 \cup_Y W_2$ be the closed, oriented union. Let $\eta\colon H_1(\partial W_1) \times H_1(\partial W_1) \to \mathbb{Z}$ be a skew-symmetric pairing, denote by Δ_η the induced variation (given by 4.1.2) and denote by $\varphi_\eta\colon W_1 \to W_1$ the induced homeomorphism (given by Theorem 4.1.7). Consider the* *umkehr map to the inclusion* $i_1 \colon W_1 \to X$,

$$i_1^! \colon H_2(X) \xrightarrow{\mathrm{PD}^{-1}} H^2(X) \xrightarrow{i_1^*} H^2(W_1) \xrightarrow{\mathrm{PD}} H_2(W_1, Y).$$

Then for any class $x \in H_2(X)$ *we have that*

$$(\varphi_\eta \cup \mathrm{Id}_{W_2})_*(x) = x - (i_1)_* \Delta_\eta(i_1^!(x)) \in H_2(X), \tag{4.1.3}$$

where $\varphi_\eta \cup \mathrm{Id}_{W_2} \colon X \to X$ *is the homeomorphism defined as* φ_η *on* W_1 *and as* Id_{W_2} *on* W_2.

Proof. Let $\Sigma \subset X$ be an embedded, closed, oriented surface representing x, transverse to Y (for topological transversality see [FQ90, Theorem 9.5A]).

The statement $i_1^!(x) = [\Sigma \cap W_1] \in H_2(W_1, Y)$ is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccc} H_2(X) & \xrightarrow{q_*} & H_2(X, W_2) & \xleftarrow{\cong} & H_2(W_1, Y) \\ {\scriptstyle \mathrm{PD}}\uparrow{\scriptstyle \cong} & & & \overset{\cong}{\nearrow}{\scriptstyle \mathrm{PD}} & \\ H^2(X) & \xrightarrow{i_1^*} & H^2(W_1) & & \end{array}$$

where q_* is the map induced by the inclusion $q \colon (X, \emptyset) \to (X, W_2)$ which sends $[\Sigma]$ to $[\Sigma \cap W_1]$ and the written isomorphism $H_2(W_1, Y) \to H_2(X, W_2)$ comes from excision.

We now prove the commutativity of the diagram. Let $y \in H^2(X)$. Going first to the right, this maps to $i_1^*(y) \frown [W_1, Y] \in H_2(W_1, Y)$ and then to $y \frown (i_1)_*[W_1, Y] \in H_2(X, W_2)$ by naturality of the (relative) cap product. Going the other way, $y \in H^2(X)$ is mapped to $q_*(y \frown [X]) = y \frown q_*[X] \in H_2(X, W_2)$, again by naturality of the (relative) cap product. It remains to be shown that these are equal, i.e. that $q_*[X] = (i_1)_*[W_1, Y]$, but this is clear by the definition of q_*. This completes the proof that the diagram commutes.

As a singular 2-chain, $\Sigma = (\Sigma \cap W_1) + (\Sigma \cap W_2) \in C_2(X)$. Similarly, the singular 2-chain induced by $(\varphi_\eta \cup \mathrm{Id}_{W_2})(\Sigma)$, i.e. the left hand side of 4.1.3, is equal to the sum $\varphi_\eta(\Sigma \cap W_1) + (\Sigma \cap W_2)$ in $C_2(X)$. Hence we have that:

$$\Sigma - (\varphi_\eta \cup \mathrm{Id}_{W_2})(\Sigma) = (\Sigma \cap W_1) - \varphi_\eta(\Sigma \cap W_1) \in C_2(X).$$

The right hand side is homologous to the cycle induced by the glued-up surface

$$(\Sigma \cap W_1) \cup -\varphi_\eta(\Sigma \cap W_1)$$

which is equal to $(i_1)_* \Delta_\eta([\Sigma \cap W_1])$ by Definition 4.1.1. $\square$

§ 4.2 | Sufficient conditions for non-smoothability

In this section, all manifolds will be considered to be smooth. For any closed, oriented, 4-manifold X, we will denote by $\mathrm{Spin}^c(X)$ the set of isomorphism classes of spin^c-

structures on X, and by $\mathcal{I}(X,\cdot)\colon \mathrm{Spin}^c(X) \to \mathcal{Y}$ a map taking values in an abelian group $\mathcal{Y}$ such that the action of $\mathrm{Diffeo}^+(X)$ on $H_2(X)$ by pull-back preserves the set of $\mathcal{I}$-basic classes, defined as

$$\mathcal{B}_{\mathcal{I}}(X) := \{c_1(\mathfrak{s}) \in H^2(X) \mid \mathcal{I}(X,\mathfrak{s}) \neq 0, \mathfrak{s} \in \mathrm{Spin}^c(X)\},$$

and moreover this set is *finite*. For example, if $b_2^+(X) \geq 2$, and $\mathfrak{s} \in \mathrm{Spin}^c(X)$, $\mathcal{I}(X,\mathfrak{s})$ may be taken to be the Seiberg-Witten invariant $\mathcal{SW}(X,\mathfrak{s})$ [Wit94], the Ozsváth-Szabó mixed invariant [OS06], or the Bauer-Furuta invariant [BF04]. The finiteness of Bauer-Furuta basic classes is not stated explicitly in [BF04], but can be proved using curvature inequalities as observed in the proof of [MMP20, Theorem 4.5]. For us, we will only be interested in taking $\mathcal{I}(X,\mathfrak{s})$ to be the Seiberg-Witten invariant, which was already introduced in Section 3.1.2.

We now prove the proposition that we will use to detect non-smoothability for homeomorphisms in the Torelli group.

Proposition 4.2.1. *Let W^4 be a compact, oriented 4-manifold with connected boundary Y. Suppose that $\pi_1(W) = 1$ and that $b_1(Y) \geq 2$. If W embeds in a* closed*, oriented 4-manifold X such that for some $\mathfrak{s} \in \mathrm{Spin}^c(X)$,*

1. *$\mathcal{I}(X,\mathfrak{s}) \neq 0$,*
2. *$i^*_{Y,X}(c_1(\mathfrak{s})) \in H^2(Y)$ is non-torsion where $i_{Y,X} : Y \hookrightarrow X$ is the inclusion,*
3. *$H^1(X \setminus W) = 0$,*

then there exists infinitely many non-smoothable mapping classes in $\mathrm{Tor}(W,Y)$. If in addition $b_1(Y) = 2$ then any non-trivial element of $\mathrm{Tor}(W,Y)$ is non-smoothable.

Proof. To avoid clutter, it is convenient to denote $\zeta_X := c_1(\mathfrak{s})$ and its restrictions by $\zeta_W := i^*_{W,X}c_1(\mathfrak{s})$ and $\zeta_Y := i^*_{Y,X}c_1(\mathfrak{s})$.

Since $\zeta_Y \in H^2(Y;\mathbb{Z})$ is not torsion $\mathrm{PD}(\zeta_Y) = dv_1$, for some $d \in \mathbb{Z} \setminus \{0\}$ and an indivisible element $v_1 \in H_1(Y;\mathbb{Z})$. Extend v_1 to $v_1, \dots, v_{b_1(Y)} \in H_1(Y)$, a lift of a basis of $H_1(Y)/\mathrm{Torsion}(H_1(Y))$. Now we set $\eta := v_1^* \wedge v_2^* \in \wedge^2(H_1(Y)^*)$ where v_i^* denotes the Hom dual with respect to the above basis (note that $v_2 \neq 0$ exists since we assumed $b_1(Y) \geq 2$).

By Theorem 4.1.7, for each $k \in \mathbb{Z}\setminus\{0\}$, there is a unique mapping class in $\mathrm{Tor}(W,Y)$ associated to $k\eta$, and we define $\varphi_k \in \mathrm{Homeo}^+(W,Y)$ to be an arbitrary representative of that class. By construction, each φ_k acts trivially on $H_2(W)$. The rest of the proof is devoted to showing that φ_k is non-smoothable for infinitely many values of k.

For each k, we define $\hat{\varphi}_k := \varphi_k \cup \mathrm{Id}_{X\setminus \mathrm{int}(W)} \in \mathrm{Homeo}^+(X)$ as in Lemma 4.1.9 to be the homeomorphism obtained by extending φ_k as the identity on $X \setminus W$. The non-smoothability of $\hat{\varphi}_k$ for infinitely many k, which we are now going to prove, implies the analogous statement for φ_k.

We will show that $\{\hat{\varphi}_k^* \zeta_X\}_{k \in \mathbb{Z}\setminus\{0\}}$ is infinite. Since $\mathcal{B}_{\mathcal{I}}(X)$ is finite, and is preserved by the action of $\mathrm{Diff}^+(X)$, we will reach a contradiction.

It follows from Lemma 4.1.9 that

$$(\hat{\varphi}_k)_*(\mathrm{PD}(\zeta_X)) = \mathrm{PD}(\zeta_X) - (i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) \in H_2(X). \tag{4.2.1}$$

From 4.1.2 we see that

$$(i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) = k \cdot (i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \partial \circ \mathrm{PD}(\zeta_W) \in H_2(X).$$

We claim that the right hand side is equal to a non-torsion element times k. This will imply the desired result by 4.2.1. We have that

$$\begin{aligned}(i_{W,X})_* \circ \Delta_{\varphi_k}(\mathrm{PD}(\zeta_W)) &= k \cdot (i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \partial \circ \mathrm{PD}(\zeta_W) \\ &= k \cdot (i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad} \circ \mathrm{PD}(\zeta_Y) \\ &= k \cdot (i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1} \circ \eta^{ad}(dv_1) \\ &= kd \cdot (i_{Y,X})_* \circ \mathrm{PD} \circ \mathrm{ev}^{-1}(v_2^*).\end{aligned}$$

Since v_2 is non-torsion and the maps ev^{-1} and PD are isomorphisms, the claim will follow if we prove that

$$(i_{Y,X})_* \colon H_2(Y) \to H_2(X)$$

is injective. Note that

$$(i_{Y,X})_* = (i_{W,X})_* \circ (i_{Y,W})_*.$$

The kernel of $(i_{Y,W})_*$ is equal to the image of $H_3(W,Y) \to H_2(Y)$ in the long exact sequence of the pair, which is trivial since $H_3(W,Y) \cong H^1(W) = 0$. Similarly the kernel of $(i_{W,X})_*$ is equal to the image of $H_3(X,W) \to H_2(W)$ which is zero because $H_3(X,W) \cong H_3(X \setminus \mathrm{int}(W), Y)$ by excision and, by assumption, $0 = H^1(X \setminus W) \cong H_3(X \setminus \mathrm{int}(W), Y)$. Being the composition of injective maps, $i_{Y,W}$ is injective. This completes the proof that there are infinitely many non-smoothable mapping classes in $\mathrm{Tor}(W,Y)$.

To prove the last statement we assume now that $b_1(Y) = 2$. Then, under the isomorphism from Theorem 4.1.7, we can identify $\mathrm{Tor}(W,Y)$ with the infinite cyclic group generated by η. Above we showed that there exists $k_0 > 0$ such that $\varphi_{k\eta}$ is non-smoothable for any $|k| > k_0$. The non-smoothability of $\mathrm{Tor}(W,Y) \setminus \{\mathrm{Id}_X\}$ follows from this by using the fact that smoothable mapping classes form a subgroup of $\mathrm{Tor}(W,Y)$ and that all non-trivial subgroups of $\mathbb{Z}$ are infinite. □

§ 4.3 | Constructing examples

In this section we will construct the infinite family of examples of non-smoothable homeomorphisms which lie in the Torelli subgroup, hence proving Theorem 4.0.1.

Let Z be the 4-manifold with boundary defined by the Kirby diagram in Fig-

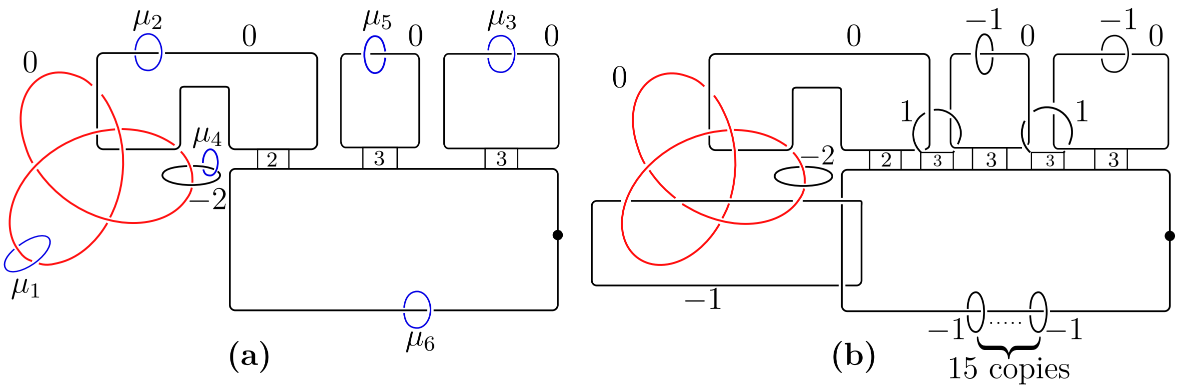


Figure 4.1: (a) Kirby diagram for the 4-manifold Z, (b) Kirby diagram showing an embedding of Z into $K_3\#2\overline{\mathbb{CP}}^2$.

ure 4.1 (a). Let $T \subset \operatorname{int}(Z)$ be the embedded torus obtained by capping the genus one Seifert surface for the red trefoil knot with the core of the handle attached along it. From the diagram it is clear that $[T] \neq 0 \in H_2(Z)$ and $[T]^2 = 0$.

For any $n \in \mathbb{N}$ we define the knot $K(n)$ to be the twist knot with Alexander polynomial $\Delta_{K(n)} = -(2n-1) + n(t + t^{-1})$, and $E_{K(n)}$ to be the knot exterior of $K(n)$ in S^3. Then we define $Z_n := E(K) \times S^1 \cup_\partial (Z \setminus \nu T)$ to be the manifold obtained by performing knot surgery Definition 3.1.5 on the torus T using the knot $K(n)$. Since the knot surgery only changes the manifold in the interior, we have an identification $\partial Z_n \cong Y := \partial Z$.

Theorem 4.3.1. *The 4-manifolds $\{Z_n\}_{n\in\mathbb{N}}$ are all homeomorphic relative to Y, but pairwise not diffeomorphic relative to Y. They are all simply-connected with intersection form $(\mathbb{Z}^2 \oplus \mathbb{Z}^2, \left[\begin{smallmatrix}0 & 1\\ 1 & -2\end{smallmatrix}\right] \oplus 0)$ and have infinite* $\operatorname{Tor}(Z_n, Y)$*. Moreover, all non-trivial elements of the Torelli group* $\operatorname{Tor}(Z_n, Y)$ *are non-smoothable.*

Proof. The torus $T \subset Z$ embeds in a Gompf nucleus $N(2)$ [Gom91] by construction and so, in particular, it is near-cusp embedded (see Definition 3.1.1). Hence, the first part of the first statement follows directly from Proposition 3.1.12. In Figure 4.1(b) we depict an embedding of Z into $X := K_3\#2\overline{\mathbb{CP}}^2$, whose Kirby diagram has been taken from [GS99, Figure 8.16] (see also [AKMR15]). Hence Z_n embeds into the closed manifold X_n obtained by performing knot surgery on $T \hookrightarrow X$ using $K(n)$. From Theorem 3.1.15 and the blow-up formula [FS95], it follows that the manifolds X_n are pairwise non-diffeomorphic. Indeed, the Seiberg-Witten invariant of X_n, seen as an element of the group ring $\mathbb{Z}[H^2(X_n)]$, is equal to

$$\mathcal{SW}(X_n) = (E_1 + E_1^{-1})(E_2 + E_2^{-1})\left(-(2n-1) + n(F^2 + F^{-2})\right), \qquad (4.3.1)$$

where $E_i \in H^2(X_n)$ are the classes coming from the two blow-ups and $F := \operatorname{PD}[T]$ is the Poincaré dual to the torus T. Now, since X_n is obtained by capping Z_n with a fixed manifold $Q := X \setminus Z$ independent from n, the manifolds Z_n are pairwise non-diffeomorphic relative to their boundaries.

It remains to prove the last statement of the theorem. We want to apply Proposition 4.2.1, so we check that the hypotheses hold. We have that $H_1(Y)$ is isomorphic to $\mathbb{Z}^2$ generated by $v_1 := \mu_2 + \mu_3$ and $v_2 := \mu_5$, where the μ_i are the meridians to the components as shown in Figure 4.1(a). From 4.3.1 we see that $E_1 + E_2 \in H_2(Z_n)$ ($E_1 E_2$ in group ring notation) is a Seiberg-Witten basic class for X_n, which restricts to $\partial \circ \mathrm{PD}(E_1 + E_2) = \mu_3 + \mu_5 = -2(v_1 + v_2)$ in Y. Moreover, the complement Q is obtained by adding only 2-handles and a single 4-handle to Y and hence $H_3(X_n, Z_n) \cong H_3(Q, Y) \cong H^1(Q) = 0$. Now the final statement of the theorem follows from Proposition 4.2.1. □

§ 4.4 | Generalised Dehn twists

§ 4.4.1 | Absolute non-smoothability and generalised Dehn twists

We begin by reviewing absolute and relative smoothability from the point of view of spaces of maps.

Let X be an compact, smooth, oriented 4-manifold with boundary ∂X. We will denote by

$$\Phi : \pi_0 \operatorname{Diff}^+(X) \to \pi_0 \operatorname{Homeo}^+(X)$$
$$\Phi_\partial : \pi_0 \operatorname{Diff}^+(X, \partial X) \to \pi_0 \operatorname{Homeo}^+(X, \partial X)$$

the induced maps on the mapping class groups (see Definition 2.2.2).

Recall the definition of (relative) smoothability (Definition 2.2.1) and absolute smoothability (Definition 2.2.5). If we let $i\colon \pi_0 \operatorname{Homeo}^+(X, \partial X) \to \pi_0 \operatorname{Homeo}^+(X)$ be the map induced by the inclusion. We can see that $\varphi \in \pi_0 \operatorname{Homeo}^+(X, \partial X)$ is absolutely non-smoothable if $i(\varphi) \in \pi_0 \operatorname{Homeo}^+(X)$ does not belong to $\operatorname{im} \Phi$.

Explicitly, the difference between a relatively and an absolutely smoothable homeomorphism is that in the latter case the isotopy at each time does not need to fix the boundary pointwise. We now aim to prove that, for 4-manifolds, the concepts of relative and absolute smoothability coincide. An important role in the proof is played by generalised Dehn twists [OP23, Section 1.2], which we now review.

Definition 4.4.1. Let X be a compact, smooth, oriented n-manifold with boundary. Given $[\gamma] \in \pi_1 \operatorname{Diff}^+(\partial X)$, we define the *generalised Dehn twist* with respect to $[\gamma]$ to be the smooth isotopy class of the diffeomorphism $\varphi_\gamma \colon X \to X$ defined on a collar of ∂X as $\varphi(y, t) = (\gamma(t)(y), t) \in (\partial X) \times I$ and extended outside of the collar as the identity map.

Another point of view is the following. The sequence of inclusion and restriction

$$\operatorname{Diff}^+(X, \partial X) \to \operatorname{Diff}^+(X) \to \operatorname{Diff}^+(\partial X)$$

and the equivalent sequence in the topological category are fibration sequences. This can be proved using results of Lashof [Las76] and this proof is carried out in [OP23,

Appendix A] in the topological case. It can be shown that the connecting morphism $\pi_1 \operatorname{Diff}^+(\partial X) \to \pi_0 \operatorname{Diff}^+(X, \partial X)$ of the long exact sequence of homotopy groups is precisely the map that associates to a loop of diffeomorphisms $[\gamma]$ its generalised Dehn twist $[\varphi_\gamma]$ [OP23, Section 1.4].

Proposition 4.4.2. *Let X be a compact, smooth, oriented 4-manifold. Then a homeomorphism $f\colon X \to X'$ is (relatively) non-smoothable if and only if it is absolutely non-smoothable.*

Proof. We will prove that relative non-smoothability implies absolute non-smoothability, the other implication is clear.

Taking the associated long exact sequences and using the map Φ (and analogies thereof) we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\pi_1 \operatorname{Diff}^+(\partial X) & \longrightarrow & \pi_0 \operatorname{Diff}^+(X, \partial X) & \xrightarrow{i} & \pi_0 \operatorname{Diff}^+(X) & \xrightarrow{\partial} & \pi_0 \operatorname{Diff}^+(\partial X) \\
\downarrow \cong & & \downarrow \Phi_\partial & & \downarrow \Phi & & \downarrow \cong \\
\pi_1 \operatorname{Homeo}^+(\partial X) & \longrightarrow & \pi_0 \operatorname{Homeo}^+(X, \partial X) & \xrightarrow{i} & \pi_0 \operatorname{Homeo}^+(X) & \xrightarrow{\partial} & \pi_0 \operatorname{Homeo}^+(\partial X)
\end{array}$$

where i denotes the maps induced by the inclusion and ∂ denotes the restriction to the boundary. We are also using here that the inclusion induced map $\operatorname{Homeo}(Y) \to \operatorname{Diff}(Y)$ is a homotopy equivalence for any 3-manifold Y [Cer59, Hat83] (we will revisit this fact in Chapter 9).

Let $\phi \in \pi_0 \operatorname{Homeo}^+(X, \partial X)$ be absolutely smoothable, i.e. assume that there exists $\psi \in \pi_0 \operatorname{Diff}^+(X)$ such that $\Phi(\psi) = i(\phi)$. Then since $\partial(i(\varphi)) = \operatorname{Id}_{\partial X}$, the commutativity and exactness of the diagram implies that there exists $\psi' \in \pi_0 \operatorname{Diff}^+(X, \partial X)$ such that $i(\psi') = \psi$. Hence $\Phi_\partial(\psi')$ is equal to ϕ modulo composition with an element in the image of $\pi_1 \operatorname{Diff}^+(\partial X) \to \pi_0 \operatorname{Homeo}^+(X, \partial X)$. Thus $\Phi_\partial(\psi') = [\varphi_\gamma] \circ \phi$ for some generalised Dehn twist $[\varphi_\gamma]$, but then $\phi = [\varphi_\gamma]^{-1} \circ \Phi_\partial(\psi')$ presents ϕ as a composition of diffeomorphisms, contradicting the non-smoothability of ϕ. □

§ 4.4.2 | Realising smoothable elements of the Torelli group by generalised Dehn twists

Since generalised Dehn twists are supported in a collar of the boundary, it is clear that these give rise to smooth elements in the Torelli group of the 4-manifold. One could ask whether generalised Dehn twists generate the whole Torelli group. The next proposition gives an answer under the assumption that the boundary is connected and prime; the general case is still unknown to the authors' best knowledge.

Proposition 4.4.3. *Let X be a smooth, compact, simply-connected, oriented 4-manifold with connected and* prime *boundary Y. Then the topological Torelli group $\operatorname{Tor}(X, Y)$ is realised by generalised Dehn twists if and only if one of the following holds:*

1. *$b_1(Y) < 2$,*

2. $b_2(Y) = 2$ *and* Y *is Seifert fibered with base orbifold* T^2*,*

3. $Y = T^3$*,*

where T^n *denotes the* n*-torus.*

Proof. We begin by showing that (1) or (3) implies that $\operatorname{Tor}(X,Y)$ is realised by generalised Dehn twists. First suppose that $b_1(Y) < 2$. Then $\wedge^2 H_1(Y)^* = 0$ and hence $\operatorname{Tor}(X,Y)$ is trivial. The case $Y = T^3$ can be handled by applying [OP23, Proposition 8.9] to the three generalised Dehn twists induced by the three S^1-factors. More precisely, let $\alpha_1, \alpha_2, \alpha_3$ be the basis of $H_1(Y)$ induced by the S^1-factors of $T^3 = S^1 \times S^1 \times S^1$ and let $\alpha_1^*, \alpha_2^*, \alpha_3^* \in H_1(Y)^*$ be the dual basis. Then the element of $\wedge^2 H_1(Y)^*$ associated to a rotation of the i-th S^1-factor is $\pm\alpha_k^* \wedge \alpha_j^*$, where $k, j \neq i$ [OP23, Proposition8.9], and hence the three rotations generate the whole Torelli group.

We now show that if neither (1) nor (3) hold, then either (2) holds or $\operatorname{Tor}(X,Y)$ is not realised by generalised Dehn twists. So assume that $Y \neq T^3$ and $b_1(Y) \geq 2$. In this case Y is Haken [Wal68, 1.1.6] (therein called *sufficiently large*), and [Hat76] implies that $\pi_1 \operatorname{Diff}(Y) \cong Z(\pi_1(Y))$, hence in particular is abelian. Then it follows from [Wal67, Satz 4.1] that either the center $Z(\pi_1(Y))$ is trivial or Y is Seifert fibered over an orientable orbifold. In the former case, $b_1(Y) \geq 2$ implies that the Torelli group, being non-trivial, cannot be generated by generalised Dehn twists. In the latter case, $Z(\pi_1(Y)) \cong \mathbb{Z}$ generated by a principal orbit of the S^1-action [Wal67], hence $\pi_1 \operatorname{Diff}(Y) \to \operatorname{Tor}(X,Y)$ cannot be surjective if $b_1(Y) > 2$, for in this case $\operatorname{Tor}(X,Y)$ has rank at least two.

We finish by showing that (2) implies that $\operatorname{Tor}(X,Y)$ is realised by generalised Dehn twists. When $b_1(Y) = 2$ and Y is Seifert fibered over an orientable orbifold, the quotient is necessarily T^2 [BLPZ03]. Moreover the variation associated to the S^1-action is computed in [OP23, Proposition 8.9] and in this case it generates the whole of $\wedge^2 H_1(Y)^* \cong \mathbb{Z}$. □

In particular, if the boundary satisfies any of the three conditions of Proposition 4.4.3 then it is impossible to find a non-smoothable homeomorphism in the Torelli group.

Given the existence of non-smoothable elements of the Torelli group, we can say more. It is possible to find smoothable elements of the Torelli group which are not isotopic to any diffeomorphism supported on a collar of the boundary, let alone are realised by generalised Dehn twists.

Theorem 4.4.4. *Let* X *be a smooth, simply-connected, oriented, compact* 4*-manifold such that there exists a non-smoothable self-homeomorphism* $\varphi \in \operatorname{Tor}(X, \partial X)$*. Then there exists an integer* $m \geq 1$ *such that*

$$\varphi \# \operatorname{Id}\colon X\#(\#^m S^2 \times S^2) \to X\#(\#^m S^2 \times S^2)$$

is a smoothable homeomorphism not isotopic to any smooth map supported on a collar of the boundary.

The proof relies on the following result, which is a direct consequence of Proposition 5.4.5, due to Freedman-Quinn. We will prove in Proposition 5.4.5 in Chapter 5, but we state its consequence for simply-connected 4-manifolds now. Note that this also uses the pseudo-isotopy implies isotopy theorem of Perron and Quinn [Per86, Qui86].

Proposition 4.4.5. *Let X be a smooth, simply-connected, compact 4-manifold with boundary and $\varphi\colon X \to X$ a self-homeomorphism. Then there exists an integer $m \geq 1$ such that*

$$\psi := \varphi \# \operatorname{Id}\colon X\#(\#^m S^2 \times S^2) \to X\#(\#^m S^2 \times S^2)$$

is isotopic to a diffeomorphism relative to the boundary, i.e. φ is stably smoothable.

Proof of Theorem 4.4.4. Let $\varphi \in \operatorname{Tor}(X, \partial X)$ be one of the non-smoothable mapping classes. By Proposition 4.4.5 there exists an integer $m \geq 1$ such that

$$\psi := \varphi \# \operatorname{Id}\colon X\#(\#^m S^2 \times S^2) \to X\#(\#^m S^2 \times S^2)$$

is isotopic to a diffeomorphism. Since ψ was defined by extending φ via the identity onto the $S^2 \times S^2$ summands, we also have that $\psi \in \operatorname{Tor}(X\#(\#^m S^2 \times S^2))$. Now assume for a contradiction that ψ is supported on a collar of $\partial X\#(\#^m S^2 \times S^2) \cong \partial X$. Then we can remove the $S^2 \times S^2$ summands and obtain a diffeomorphism $\psi'\colon X \to X$. However, since we have an identification

$$\operatorname{Tor}(X, \partial X) \cong \operatorname{Tor}(X\#(\#^m S^2 \times S^2))$$

we can see that $\Delta_{\psi'} = \Delta_{\varphi}$. Hence, by Theorem 4.1.7 we see that ψ' and φ must be isotopic relative to the boundary. This contradicts the assumption that φ was not isotopic to a diffeomorphism, and so we conclude that ψ is not isotopic to any smooth map supported on a collar of the boundary. □

We now obtain Theorem 4.0.2 as a corollary.

Proof. Note that $\operatorname{Tor}(Z_n, Y) \cong \mathbb{Z}$, since $H_1(Y) \cong \mathbb{Z}^2$. Let φ_n denote the generator of $\operatorname{Tor}(Z_n, Y)$. Applying Theorem 4.4.4 to φ_n gives a diffeomorphism

$$\psi_n\colon Z_n^\dagger \xrightarrow{\cong} Z_n^\dagger$$

where $Z_n^\dagger \cong \mathbb{Z}_n\#(\#^{k_n} S^2 \times S^2)$ with the integer k_n only depending on n. By the identification $\operatorname{Tor}(Z_n, Y) \cong \operatorname{Tor}(Z_n^\dagger, Y)$, we see that ψ_n generates $\operatorname{Tor}(Z_n^\dagger, Y)$ and hence all of $\operatorname{Tor}(Z_n^\dagger, Y)$ is smoothable but only the trivial element can be realised by a diffeomorphism supported away on a collar of the boundary. □

Chapter 5

The Casson-Sullivan invariant

In this section we define the Casson-Sullivan invariant and prove some of its fundamental properties. The Casson-Sullivan invariant is named for Andrew Casson and Dennis Sullivan, and the canonical reference is the collection of papers edited by Andrew Ranicki [RCS+96].

§ 5.0.1 | Chapter outline

We begin in Section 5.1 by stating the relevant theory about microbundles and classifying spaces which we will need to define the invariant in Section 5.2. In Section 5.3 we investigate to what extent the Casson-Sullivan invariant of a homeomorphism depends on the choice of smooths structures on the manifolds involved. We then establish the Casson-Sullivan invariant's fundamental properties in Section 5.4, before proving a "connected-sum along a circle" formula (Theorem 5.5.3) for the invariant in Section 5.5.

§ 5.1 | Microbundles and classifying spaces

Let $\mathrm{TOP}(k) = \{g \colon \mathbb{R}^k \xrightarrow{\approx} \mathbb{R}^k | g(0) = 0\}$ and let $\mathrm{O}(k)$ be the group of orthogonal k-dimensional matrices. Then there are obvious inclusions $\mathrm{TOP}(k) \hookrightarrow \mathrm{TOP}(k+1)$ and $\mathrm{O}(k) \hookrightarrow \mathrm{O}(k+1)$, and we denote the corresponding direct limits as TOP and O, respectively. We will use the notation CAT to stand in for TOP and O. The classifying spaces $\mathcal{B}\mathrm{TOP}$ and $\mathcal{B}\mathrm{O}$ then classify stable $\mathbb{R}^n$ fibre bundles and stable vector bundles, respectively. The universal stable vector bundle has an underlying stable topological $\mathbb{R}^n$ fibre bundle and its classifying map will be denoted as $\xi \colon \mathcal{B}\mathrm{O} \to \mathcal{B}\mathrm{TOP}$. Similarly, let $\mathcal{B}_{\mathrm{DIFF}}$ and $\mathcal{B}_{\mathrm{TOP}}$ denote the classifying spaces of stable DIFF-microbundles and stable TOP-microbundles, respectively (see [KS77, Essay IV, §10]), and let ξ' denote the classifying map of the universal DIFF-microbundle's underlying TOP-microbundle.

Lemma 5.1.1. *Let* CAT *stand in for* TOP *or* O*. Let* $u_{\mathrm{CAT}} \colon \mathcal{B}\mathrm{CAT} \to \mathcal{B}_{\mathrm{CAT}}$ *denote the classifying map of the universal stable* CAT*-bundle's underlying stable* CAT*-microbundle. Then* u_{CAT} *is a homotopy equivalence.*

Proof. The Kister-Mazur theorem [Kis64, KL66, SGH73] gives that isomorphism classes of stable CAT bundles over a CW-complex X are in one-to-one correspondence with isomorphism classes of stable CAT-microbundles over X. This means that there is a natural bijection $\kappa_{\mathrm{CAT}}\colon [X, \mathcal{B}\mathrm{CAT}] \leftrightarrow [X, \mathcal{B}_{\mathrm{CAT}}]$ defined as the composition of the natural bijections

$$[X, \mathcal{B}\mathrm{CAT}] \to \mathrm{CAT}(X) \to \mathrm{Mic}_{\mathrm{CAT}}(X) \to [X, \mathcal{B}_{\mathrm{CAT}}]$$

where $\mathrm{CAT}(X)$ denotes the set of isomorphism classes of stable CAT-bundles over X (i.e. stable $\mathbb{R}^n$ fibre bundles if CAT = TOP and stable vector bundles if CAT = DIFF) and $\mathrm{Mic}_{\mathrm{CAT}}(X)$ denotes the set of isomorphism classes of stable CAT-microbundles over X. In fact, by the definition we can conclude that $\kappa_{\mathrm{CAT}} = (u_{\mathrm{CAT}})_*$. Then naturality gives the following commutative diagram.

$$\begin{array}{ccc} [\mathcal{B}\mathrm{CAT}, \mathcal{B}\mathrm{CAT}] & \xrightarrow{(u_{\mathrm{CAT}})_*} & [\mathcal{B}\mathrm{CAT}, \mathcal{B}_{\mathrm{CAT}}] \\ {\scriptstyle (u_{\mathrm{CAT}})^*}\big\uparrow & & {\scriptstyle (u_{\mathrm{CAT}})^*}\big\uparrow \\ [\mathcal{B}_{\mathrm{CAT}}, \mathcal{B}\mathrm{CAT}] & \xrightarrow{(u_{\mathrm{CAT}})_*} & [\mathcal{B}_{\mathrm{CAT}}, \mathcal{B}_{\mathrm{CAT}}] \end{array}$$

By a simple diagram chase one can see that there exists an element $f \in [\mathcal{B}_{\mathrm{CAT}}, \mathcal{B}\mathrm{CAT}]$ such that $(u_{\mathrm{CAT}})^*(f) = \mathrm{Id}_{\mathcal{B}\mathrm{CAT}}$ and such that $(u_{\mathrm{CAT}})_*(f) = \mathrm{Id}_{\mathcal{B}_{\mathrm{CAT}}}$. Hence f is the homotopy inverse of u_{CAT}, completing the proof. □

Now replace $\mathcal{B}\mathrm{O}$ and $\mathcal{B}_{\mathrm{DIFF}}$ by homotopy equivalent spaces (which we still denote in the same manner) such that ξ and ξ' become fibrations. We have the following square which is a homotopy pullback

$$\begin{array}{ccc} \mathcal{B}\mathrm{O} & \xrightarrow{u_{\mathrm{DIFF}}} & \mathcal{B}_{\mathrm{DIFF}} \\ \big\downarrow{\scriptstyle \xi} & & \big\downarrow{\scriptstyle \xi'} \\ \mathcal{B}\mathrm{TOP} & \xrightarrow{u_{\mathrm{TOP}}} & \mathcal{B}_{\mathrm{TOP}} \end{array}$$

and it follows that the fibres of ξ and ξ' are homotopy equivalent. We will denote this space as TOP/O. Boardman-Vogt [BV68] showed that we can 'deloop' this fibre to obtain a space $\mathcal{B}(\mathrm{TOP/O})$ and that we can extend ξ to the right to obtain the fibration sequence

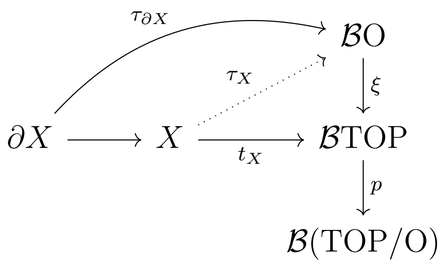
$$\mathrm{TOP/O} \to \mathcal{B}\mathrm{O} \xrightarrow{\xi} \mathcal{B}\mathrm{TOP} \xrightarrow{p} \mathcal{B}(\mathrm{TOP/O}).$$

By the above exposition, we can dispense with considering $\mathcal{B}_{\mathrm{TOP}}$ and $\mathcal{B}_{\mathrm{DIFF}}$ and we will consider the classifying map of a TOP-microbundle to be a map to $\mathcal{B}\mathrm{TOP}$ and the classifying map of a DIFF-microbundle to be a map to $\mathcal{B}\mathrm{O}$. This is helpful as much is known about the homotopy types of $\mathcal{B}\mathrm{O}$, $\mathcal{B}\mathrm{TOP}$ and TOP/O, whereas it is useful to work solely with CAT-microbundles, rather than passing between microbundles and vector bundles depending on the category.

§ 5.2 | Definition of the Casson-Sullivan invariant

Let X be an n-dimensional topological manifold with (potentially empty) boundary ∂X. Further assume that we already have a smooth structure on the boundary ∂X.

The manifold X admits a stable topological tangent microbundle $t_X\colon X \to \mathcal{B}\mathrm{TOP}$. The first obstruction for X to be smoothable is to be able to lift t_X to a stable DIFF-microbundle $\tau_X\colon X \to \mathcal{B}\mathrm{O}$ which extends the lift $\tau_{\partial X}$ which is already defined on ∂X, which we have since we already supposed the existence of a smooth structure on the boundary. We have the following diagram



It is known [KS77, Essay IV, §10.12] that there is a 6-connected map $\mathrm{TOP}/\mathrm{O} \to K(\mathbb{Z}/2,3)$, and this means we can consider the composite $p \circ t_X$ as a map $X \to K(\mathbb{Z}/2,4)$ since X is 4-dimensional. The lift $\tau_{\partial X}$ corresponds to a null-homotopy $h_t(\tau_{\partial X})$ of $p \circ (t_X|_{\partial X})$ and, since the inclusion $\partial X \to X$ is a cofibration, this homotopy extends to a homotopy

$$\widetilde{h}_t(\tau_{\partial X})\colon X \times I \to \mathcal{B}(\mathrm{TOP}/\mathrm{O})$$

such that $\widetilde{h}_0(\tau_{\partial X}) = p \circ t_X$ and such that $\widetilde{h}_1(\tau_{\partial X})$ defines an element

$$\widetilde{h}_1(\tau_{\partial X}) \in [(X, \partial X), (K(\mathbb{Z}/2,4), *)] \cong H^4(X, \partial X; \mathbb{Z}/2).$$

Definition 5.2.1. Let $(X, \partial X)$ be as above. We define the *Kirby-Siebenmann invariant*

$$\mathrm{ks}(X, \partial X) := \widetilde{h}_1(\tau_{\partial X}) \in H^4(X, \partial X; \mathbb{Z}/2).$$

Theorem 5.2.2 ([KS77, Essay IV, §10], [FQ90, Corollary 8.3D])**.** *The stable tangent microbundle of X^n for $4 \leq n \leq 7$, written as $t_X\colon X \to \mathcal{B}\mathrm{TOP}$, lifts to a stable tangent bundle τ_X extending the already specified lift on ∂X if and only if* $\mathrm{ks}(X) = 0 \in$ *$H^4(X, \partial X; \mathbb{Z}/2)$.*

Remark 5.2.3. In the high-dimensional case ($n \geq 5$) this is the first in a sequence of obstructions (and for $n = 5, 6, 7$ it is the only one), and the vanishing of all of these obstructions implies the existence of a smooth structure on X extending the given one on ∂X. However, for $n = 4$ we do not have the corresponding geometric outcome if $\mathrm{ks}(X) = 0$. For example, the manifold $E_8 \# E_8$ has vanishing Kirby-Siebenmann invariant but does not admit a smooth structure. Instead, one gets that $\mathrm{ks}(X) = 0$ implies that there exists some $k \geq 0$ such that $X \#^k (S^2 \times S^2)$ admits a smooth structure [FQ90, Section 8.6].

Remark 5.2.4. The Kirby-Siebenmann invariant is usually defined as the obstruction to lifting the stable TOP-microbundle to a stable PL-microbundle, where PL denotes the piecewise linear category. However, in the dimensions that we will examine there is no difference between the DIFF and PL categories, so we will ignore this. Indeed, there is a 6-connected map from TOP/O to TOP / PL, and TOP / PL is homotopy equivalent to $K(\mathbb{Z}/2, 3)$. We will say no more about the PL category in this thesis.

Definition 5.2.5. Let X be a 4-manifold with (potentially empty) boundary ∂X. We say that X is *formally smoothable* if there exists a lift of the stable tangent microbundle $t_X\colon X \to \mathcal{B}\mathrm{TOP}$ to a stable DIFF-microbundle $\tau_X\colon X \to \mathcal{B}\mathrm{O}$ extending the already specified lift on ∂X. We call any such lift a *formal smooth structure*. Equivalently (by Theorem 5.2.2), X is formally smoothable if the Kirby-Siebenmann invariant $\mathrm{ks}(X) = 0 \in H^4(X, \partial X; \mathbb{Z}/2)$. We say that X is *formally smooth* if it is equipped with a choice of lift τ_X.

Remark 5.2.6. A smooth structure on a topological manifold determines a canonical formal smooth structure after choosing a Riemannian metric on the manifold. This is proved by first showing that after choosing a Riemannian metric we have a canonical bundle isomorphism between a smooth manifold's tangent bundle and its normal bundle in the diagonal embedding (see [MS74, Lemma 11.5]), and then observing that the normal bundle of the diagonal embedding's underlying topological microbundle has a canonical microbundle isomorphism to the tangent microbundle of the manifold. Since the space of Riemannian metrics on a smooth manifold is contractible, this means that a smooth structure on a topological manifold determines an essentially unique formal smooth structure.

We wish to define the Casson-Sullivan invariant as the Kirby-Siebenmann invariant of the mapping cylinder of a homeomorphism, but there is a subtlety that must be addressed first. Let $f\colon X \to X'$ be a homeomorphism of (formally) smooth manifolds. Let t_X, $t_{X'}$ denote the classifying maps of the TOP-microbundles of X and X' and let τ_X and $\tau_{X'}$ denote their corresponding lifts to $\mathcal{B}\mathrm{O}$ given by their (formal) smooth structures. The pullback $f^*(\tau_{X'}) = \tau_{X'} \circ f$ is not a lift of t_X immediately (rather, it is a lift of $t_{X'} \circ f$), but there is a homotopy which is unique up to homotopy to make it a lift of t_X.

Lemma 5.2.7. *There is a homotopy $h(f)_t\colon X \times I \to \mathcal{B}\mathrm{TOP}$ such that $h(f)_0 = t_{X'} \circ f$ and $h(f)_1 = t_X$, which is unique up to homotopy.*

Proof. The homeomorphism f induces a canonical microbundle isomorphism between the microbundles t_X and the pullback bundle $f^*(t_X)$. Use this isomorphism to form the mapping cylinder of microbundles on $X \times I$ and denote this by $\mathfrak{X}(f)$. By Kirby-Siebenmann [KS77, Essay IV, Proposition 8.1], TOP-microbundles over $X \times I$ which restrict to t_X and $f^*(t_X)$ on either end are in one-to-one correspondence with homotopy classes of maps $X \times I \to \mathcal{B}\mathrm{TOP}$ restricting to the classifying maps t_X and $t_{X'} \circ f$. Since $\mathfrak{X}$ is such a TOP-microbundle, we get a well-defined up to (relative) homotopy map $h'(f)\colon X \times I \to \mathcal{B}\mathrm{TOP}$ such that $h'(f)_0 = t_X$ and $h'(f)_1 = t_{X'} \circ f$, i.e. a homotopy between these classifying maps that is well-defined up to (relative) homotopy. Taking the reverse homotopy gives the desired homotopy from $t_{X'} \circ f$ to t_X.

To get uniqueness of this homotopy up to homotopy, one can again use Kirby-Siebenmann [KS77, Essay IV, Proposition 8.1] on $(X \times I) \times I$, and then use the same argument as above. □

Since the map $\xi\colon \mathcal{B}\mathrm{O} \to \mathcal{B}\mathrm{TOP}$ is a fibration, we can lift the homotopy $h(f)_t$ from Lemma 5.2.7 to a homotopy $\widetilde{h}(f)_t$ such that $\widetilde{h}(f)_0 = \tau_{X'} \circ f = f^*(\tau_{X'})$ and such that $\widetilde{h}(f)_1$ is a lift of t_X. Since this homotopy is unique up to homotopy, we will cease to mention this and instead whenever we write $f^*(\tau_{X'})$ it should be taken to mean that we have homotoped this such that it is a lift of t_X (in the unique way). In fact, one can extend the argument in the proof of Lemma 5.2.7 to see that all higher homotopies are unique up to homotopy, etc., and hence there is an essentially unique classifying map for the tangent microbundle. We will not need this stronger statement.

Remark 5.2.8. Lemma 5.2.7 gives us immediately that the Kirby-Siebenmann invariant is natural with respect to homeomorphisms. More precisely, it implies that for a homeomorphism of topological manifolds $f\colon X \to X'$ we have that $\mathrm{ks}(X) = f^* \mathrm{ks}(X')$, since the homotopy classes of maps representing these cohomology classes are clearly homotopic by Lemma 5.2.7. By an analogous argument, the Kirby-Siebenmann invariant is also natural with respect to inclusion of open submanifolds (see the last part of [KS77, Essay IV, Theorem 10.1]).

We now define the Casson-Sullivan invariant.

Definition 5.2.9. Let X and X' be n-dimensional (formally) smooth manifolds with (potentially empty) boundaries $\partial X \cong \partial X'$ and let $f\colon X \to X'$ be a homeomorphism restricting to a fixed diffeomorphism on the boundary. Let M_f be the mapping cylinder

$$M_f := \frac{(X \times I) \sqcup X'}{(\{x\} \times \{1\}) \sim f(x)}$$

and note that $\tau_X \cup_\partial f^*(\tau_X)'$ defines a lift of t_{M_f} on ∂M_f. The reduced suspension (of pairs) construction gives an isomorphism

$$\varpi\colon H^3(X, \partial X; \mathbb{Z}/2) \xrightarrow{\cong} H^4(M_f, \partial M_f; \mathbb{Z}/2). \tag{5.2.1}$$

We then define the *Casson-Sullivan invariant* $\mathrm{cs}(f)$ as

$$\mathrm{cs}(f) := \varpi^{-1}(\mathrm{ks}(M_f, \partial M_f)) \in H^3(X, \partial X; \mathbb{Z}/2).$$

The isomorphism ϖ has other constructions. For example, by Poincaré duality we have $H^3(X, \partial X; \mathbb{Z}/2) \cong H_{n-3}(X; \mathbb{Z}/2)$, and by the homotopy equivalence given by the inclusion $X \hookrightarrow M_f$, we have that $H_{n-3}(X; \mathbb{Z}/2) \cong H_{n-3}(M_f; \mathbb{Z}/2)$. Poincaré duality inverse then gives an isomorphism $H_{n-3}(M_f; \mathbb{Z}/2) \cong H^4(M_f, \partial M_f; \mathbb{Z}/2)$. Composing all of these isomorphisms gives ϖ. It can also be identified with a connecting homomorphism in the appropriate long exact sequence of a triple. We will not need either of these alternate definitions.

Remark 5.2.10. It will often be useful for us to think of $\mathrm{cs}(f)$ instead as the element

$$\varpi(\mathrm{cs}(f)) \in H^4(M_f, \partial M_f; \mathbb{Z}/2).$$

In the rest of this thesis, we will reserve the symbol ϖ to always mean the isomorphism given in Equation (5.2.1) so that is clear when we are going between these two elements.

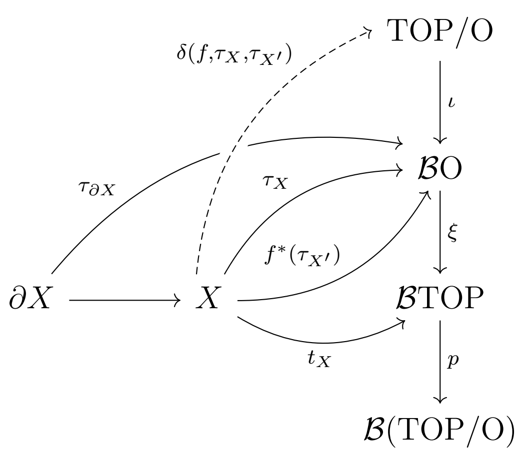
Remark 5.2.11. We now explicitly describe the homotopy class of the map corresponding to the Casson-Sullivan invariant in the following way which will be useful for proofs. Take the map $p \circ \tau_{X\times I}$ and then use the null-homotopies defined by τ_X and $f^*(\tau'_X)$ (glued over the null-homotopy for $\partial X \times I$) to construct a relative map $(X \times I, \partial(X \times I)) \to (\mathcal{B}(\mathrm{TOP/O}), *)$, which is relatively homotopic to $\varpi\,\mathrm{cs}(f)$ by construction. We will refer to this relative class as the homotopy class of $p \circ \tau_{X\times I}$ extended by the null-homotopies τ_X and $f^*\tau_X$.

In this thesis we will generally only consider homeomorphisms of 4-manifolds. In this case it is clear by Poincaré duality that the Casson-Sullivan invariant vanishes for all homeomorphisms if X is simply-connected.

§ 5.3 | Dependence of the Casson-Sullivan invariant on smooth structures

This subsection is devoted to describing the extent to which the Casson-Sullivan invariant depends on the choice of a (formal) smooth structure. We will begin by giving a different definition for the Casson-Sullivan invariant which more naturally exhibits it as an element of $H^3(X, \partial X; \mathbb{Z}/2)$.

Let $f\colon X \to X'$ be a homeomorphism and let X be a smoothable 4-manifold with boundary ∂X. For any choice of lifts τ_X, $\tau_{X'}$ of the topological tangent bundles t_X, $t_{X'}$ we have the following (very non-commutative) diagram, augmented from Section 5.2.



We will now work to define $\delta(f, \tau_X, \tau_{X'})$ in the diagram, but roughly one should think of it as the 'difference' between the lifts τ_X and $f^*(\tau_{X'})$. First, note that we have an action of $[X, \partial X; \mathrm{TOP/O}]$ on homotopy classes of lifts of the stable tangent microbundle to $\mathcal{B}\mathrm{O}$, defined using the H-space structure on $\mathcal{B}\mathrm{O}$ which we will denote by the symbol $\oplus$.

Lemma 5.3.1. *Let $(X, \partial X)$ be a topological 4-manifold with boundary and stable tangent microbundle t_X. There is a well-defined action of $[X, \partial X; \mathrm{TOP/O}]$ on lifts of the stable tangent microbundle to $\mathcal{B}\mathrm{O}$ defined by*

$$\delta \cdot \tau_X := \tau_X \oplus \iota\delta$$

and furthermore this action is free and transitive.

Proof. Let $\tau_X \colon X \to \mathcal{B}\mathrm{O}$ be a lift of t_X and let $[X, \partial X; \mathcal{B}\mathrm{O}, *]_{t_X}$ denote homotopy classes of lifts of t_X (i.e. lifts of t_X up to homotopy through lifts of t_X). By [Bau77, Theorem 1.3.8], and the surrounding discussion, there is a bijection

$$[X, \partial X; \mathrm{TOP/O}] \to [X, \partial X; \mathcal{B}\mathrm{O}, *]_{t_X}$$

given by $\delta \mapsto \tau_X \oplus \iota\delta$. This induces an action via the H-space structure on TOP/O and it being free and transitive follows by the above map being a bijection. □

We can now define $\delta(f, \tau_X, \tau_{X'})$. By Lemma 5.3.1 the lifts τ_X and $f^*(\tau_{X'})$ determine a unique element which we denote as $\delta(f, \tau_X, \tau_{X'})$ in $[X, \partial X; \mathrm{TOP/O}]$ such that $\delta(f, \tau_X, \tau_{X'}) \cdot \tau_X = f^*(\tau_{X'})$.

Proposition 5.3.2. *Let X be a formally smoothable topological 4-manifold and $f \colon X \to X'$ a homeomorphism. Let τ_X and $\widetilde{\tau}_X$ be two lifts of the stable tangent microbundle of X, let $\tau_{X'}$ and $\widetilde{\tau}_{X'}$ be two lifts of the stable tangent microbundle of X', let $a \in [X, \partial X; \mathrm{TOP/O}]$ be the unique element given by Lemma 5.3.1 such that $a \cdot \tau_X = \widetilde{\tau}_X$ and similarly let b be such that $b \cdot \tau_{X'} = \widetilde{\tau}_{X'}$. Then*

$$a + \delta(f, \widetilde{\tau}_X, \widetilde{\tau}_{X'}) = f^*(b) + \delta(f, \tau_X, \tau_{X'}).$$

Hence it follows that if $X = X'$, $\tau_X = \tau'_X$, $\widetilde{\widetilde{\tau_X}} = \widetilde{\tau}_{X'}$, and f acts trivially on $H^3(X, \partial X; \mathbb{Z}/2)$ then $\delta(f)$ is defined and does not depend on the choice of lift of the stable tangent microbundle of X.

Proof. We have that

$$(a + \delta(f, \widetilde{\tau}_X, \widetilde{\tau}_{X'})) \cdot \tau_X = \delta(f, \widetilde{\tau}_X, \widetilde{\tau}_{X'}) \cdot (a \cdot \tau_X) = \delta(f, \widetilde{\tau}_X, \widetilde{\tau}_{X'}) \cdot \widetilde{\tau}_X = f^*\widetilde{\tau}_{X'},$$

where the first equality is given by using the action, the second from the definition of a, and the third from the definition of $\delta(f, \widetilde{\tau}_X, \widetilde{\tau}_{X'})$. We also have that

$$(f^*(b)+\delta(f, \tau_X, \tau_{X'}))\cdot\tau_X = f^*(b)\cdot(\delta(f, \tau_X, \tau_{X'})\cdot\tau_X) = f^*(b)\cdot f^*\tau_{X'} = f^*(b\cdot\tau_{X'}) = f^*\widetilde{\tau}_{X'},$$

where the first equality is again given by using the action, the second from the definition of $\delta(f, \tau_X, \tau_{X'}) \cdot \tau_X)$, the third by the fact that the H-space structure is compatible with taking pre-composition by the homeomorphism f, and the final equality comes from the definition of b. Since the action of $[X, \mathrm{TOP/O}]$ is free by Lemma 5.3.1, this completes the proof. □

It remains to be seen that this definition for $\delta(f)$ matches up with the definition of the Casson-Sullivan invariant, given as the Kirby-Siebenmann invariant of the mapping cylinder.

Proposition 5.3.3. *Let X and X' be formally smooth topological 4-manifolds with formal smooth structures τ_X and $\tau_{X'}$, respectively. Furthermore, let $f\colon X \to X'$ be a homeomorphism. Then $\delta(f, \tau_X, \tau_{X'}) = \mathrm{cs}(f)$.*

Proof. By Remark 5.2.11 we can consider the Casson-Sullivan invariant as a relative homotopy class $\mathrm{cs}(f) \in [(\Sigma X_+, \{\mathrm{pt}\}), \mathcal{B}(\mathrm{TOP/O})]$, where Σ denotes the reduced suspension and X_+ denotes X with a disjoint basepoint added. By the loopspace-suspension adjoint (and that $\mathcal{B}(\mathrm{TOP}\,/\,\mathrm{O})$ is the delooping of $\mathrm{TOP}\,/\,\mathrm{O}$), this gives an element $\mathrm{cs}(f) \in [(X_+, \{\mathrm{pt}\}), \mathrm{TOP/O}]$. By construction, this element sends τ_X to $f^*(\tau_{X'})$ under the action of Lemma 5.3.1, completing the proof. □

If we put together the previous two propositions, we immediately derive the following corollary.

Corollary 5.3.4. *Let X and X' be topological manifolds each with two (formal) smooth structures $\mathscr{S}^1_X$ and $\mathscr{S}^2_X$, and $\mathscr{S}^1_{X'}$ and $\mathscr{S}^2_{X'}$, respectively. Denote the corresponding lifts of the stable tangent microbundle by τ_X and $\widetilde{\tau}_X$, and $\tau_{X'}$ and $\widetilde{\tau}_{X'}$, respectively. Let $f\colon X \to X'$ be a homeomorphism, and let $\mathrm{cs}(f, \mathscr{S}^1_X, \mathscr{S}^1_{X'})$ and $\mathrm{cs}(f, \mathscr{S}^2_X, \mathscr{S}^2_{X'})$ denote the Casson-Sullivan invariants of f with respect to the corresponding (formal) smooth structures. Let $a \in [X, \partial X; \mathrm{TOP/O}]$ be the unique element given by Lemma 5.3.1 such that $a \cdot \tau_X = \widetilde{\tau}_X$, and similarly let b be the unique element such that $b \cdot \tau_{X'} = \widetilde{\tau}_{X'}$. Then we have that*

$$a + \mathrm{cs}(f, \mathscr{S}^1_X, \mathscr{S}^1_{X'}) = f^*(b) + \mathrm{cs}(f, \mathscr{S}^2_X, \mathscr{S}^2_{X'}).$$

Proof. The proof is immediate from Proposition 5.3.2 and Proposition 5.3.3. □

If we only consider self-homeomorphisms, then $\tau_X = \tau_{X'}$, $\widetilde{\tau}_X = \widetilde{\tau}_{X'}$ and $a = b$, giving the formula

$$a + \mathrm{cs}(f, \mathscr{S}^1_X) = f^*(a) + \mathrm{cs}(f, \mathscr{S}^2_X).$$

This means, for f a self-homeomorphism, we can define $\mathrm{cs}(f)$ not only for smooth 4-manifolds, but also for *smoothable* manifolds in the case where all self-homeomorphisms must act trivially on $H^3(-;\mathbb{Z}/2)$. For example, this occurs if the fundamental group is cyclic. In fact, in these cases one could also define $\mathrm{cs}(f)$ for non-smoothable manifolds by first removing a point, and then using the fact that all topological 4-manifolds admit a smooth structure away from a point [FQ90, Section 8.2]. We will not pursue this in the rest of the thesis, and instead simplify matters by only considering our manifolds to be smooth.

§ 5.4 | Properties

We now establish properties of the Casson-Sullivan invariant. We begin by showing that it is a pseudo-isotopy invariant.

Proposition 5.4.1. *Let X and X' be smooth manifolds and let $f, g\colon X \to X'$ be a pair of homeomorphisms.*

1. *If f is pseudo-smoothable then* $\mathrm{cs}(f) = 0$.
2. *If f is pseudo-isotopic to g then* $\mathrm{cs}(f) = \mathrm{cs}(g)$.

Proof. We begin with the proof of (1). Let $\mathscr{S}_X$ and $\mathscr{S}_{X'}$ denote the smooth structures on X and X', respectively. Note that $W(f)$ is homeomorphic to $X \times I$ but with the additional specified smooth structures on the boundary, so $\mathrm{cs}(f) = 0$ if we can find a smooth structure on $X \times I$ which restricts to the given smooth structures $\mathscr{S}_X$ and $f^*(\mathscr{S}_{X'})$ on the two boundary pieces (glued along the product smooth structure $\mathscr{S}_{\partial X \times I}$). Let F be the hypothesised pseudo-isotopy between f and the diffeomorphism and denote this diffeomorphism by $\widetilde{f}$. Use the pseudo-isotopy F to pull back the product structure $\mathscr{S}_{X'} \times I$ to $F^*(\mathscr{S}_X \times I)$ on $X \times I$. We then have that $F^*(\mathscr{S}_X \times I)|_{X\times\{0\}} = (\widetilde{f})^*(\mathscr{S}_{X'}) = \mathscr{S}_X$ since $\widetilde{f}$ is a diffeomorphism and $F^*(\mathscr{S}_X \times I)|_{X\times\{1\}} = f^*(\mathscr{S}_{X'})$ and together these mean that $\mathrm{cs}(f) = 0$.

Now we prove (2). As in Remark 5.2.11, $\varpi\,\mathrm{cs}(f)$ is the homotopy class of $p \circ \tau_{X\times I}$ extended by the null-homotopies given by $\mathscr{S}_X$ and $f^*(\mathscr{S}_{X'})$. Similarly, $\varpi\,\mathrm{cs}(g)$ is the homotopy class of $p\circ\tau_{X\times I}$ extended by the null-homotopies given by $\mathscr{S}_X$ and $g^*(\mathscr{S}_{X'})$. Let F be the pseudo-isotopy from g to f. Then $F^*(\mathscr{S}_{X'} \times I)$ describes a null-homotopy of $p \circ \tau_{X\times I}$ extended by the null-homotopies given by $f^*(\mathscr{S}_{X'})$ and $g^*(\mathscr{S}_{X'})$, i.e. a homotopy between the null-homotopies given by $f^*(\mathscr{S}_X)$ and $g^*(\mathscr{S}_X)$. This gives us a homotopy relative to the boundary between the relative homotopy classes defining $\varpi\,\mathrm{cs}(f)$ and $\varpi\,\mathrm{cs}(g)$, and hence $\mathrm{cs}(f) = \mathrm{cs}(g)$. □

We can say more in the case of self-homeomorphisms. Let $\widetilde{\pi}_0\,\mathrm{Homeo}(X, \partial X)$ denote the pseudo-mapping class group of X relative to ∂X, i.e. the quotient of $\mathrm{Homeo}(X, \partial X)$ by those homeomorphisms which are pseudo-isotopic to the identity (see Definition 2.2.2 and Remark 2.2.3). Then we have the following result.

Proposition 5.4.2. *Let X be a smooth 4-manifold. Then the map*

$$\mathrm{cs}\colon \widetilde{\pi}_0\,\mathrm{Homeo}(X, \partial X) \to H^3(X, \partial X; \mathbb{Z}/2)$$

sending a representative of a pseudo-isotopy class of self-homeomorphisms to its Casson-Sullivan invariant is a crossed homomorphism. In other words, if $f, g\colon X \to X$ are *representatives of pseudo-isotopy classes of self-homeomorphisms then*

$$\operatorname{cs}(g \circ f) = \operatorname{cs}(f) + f^* \operatorname{cs}(g).$$

Proof. We first note that the well-definedness of the above map follows directly from Proposition 5.4.1. It suffices to show that for two self-homeomorphisms $f, g\colon X \to X$, we have that $\operatorname{cs}(g \circ f) = \operatorname{cs}(f) + f^* \operatorname{cs}(g)$.

We begin by giving a method for describing the group operation on

$$[X \times I, \partial(X \times I); K(\mathbb{Z}/2, 4)].$$

The normal way to define this is by using the highly-connected map $\mathcal{B}(\mathrm{TOP}/\mathrm{O}) \to \Omega K(\mathbb{Z}/2, 5)$ and then using composition of loops to define the group operation (see [Hat02, Section 4.3]). However, we also have that

$$[(X \times I, \partial); (K(\mathbb{Z}/2, 4), *)] = [\Sigma X_+, \{\mathrm{pt}\}; K(\mathbb{Z}/2, 4), *],$$

where $X_+ := X \sqcup \{\mathrm{pt}\}$ and ΣX_+ denotes the reduced suspension of X_+. Hence, $[(X \times I, \partial); K(\mathbb{Z}/2, 4)]$ has a natural operation by 'stacking' cylinders given by the natural group operation coming from the reduced suspension. By the suspension-loopspace adjoint relation,

$$[\Sigma X_+, \{\mathrm{pt}\}; K(\mathbb{Z}/2, 4), *] = [X_+, \{\mathrm{pt}\}; \Omega K(\mathbb{Z}/2, 4), *] = [X_+, \{\mathrm{pt}\}; \Omega\Omega K(\mathbb{Z}/2, 5), *]$$

and the group operation given by 'stacking' corresponds to the group operation given by the outermost loopspace on the right. The standard group operation comes from the inner loopspace structure, but these two are equivalent.

Let $\mathscr{S}_X$ denote the smooth structure on X. Consider the homotopy class of maps corresponding to $\varpi(\operatorname{cs}(f) + f^* \operatorname{cs}(g)) = \varpi \operatorname{cs}(f) + (f \times \mathrm{Id})^* \varpi \operatorname{cs}(g)$. (Here we used the definition of ϖ from Definition 5.2.9). We again use the terminology in Remark 5.2.11. This corresponds to stacking the homotopy class of $p \circ \tau_{X \times I}$ extended by the null-homotopies given by $\mathscr{S}_X$ and $f^*(\mathscr{S}_X)$ with the homotopy class of $p \circ \tau_{X \times I}$ extended by the null-homotopies given by $f^*(\mathscr{S}_X)$ and $(g \circ f)^*(\mathscr{S}_X)$. But this is relatively homotopic to the homotopy class of $p \circ \tau_{X \times I}$ extended by the null-homotopies given by $\mathscr{S}_X$ and $(g \circ f)^*(\mathscr{S}_X)$. The relative homotopy is given by the null-homotopy corresponding to $(f \times \mathrm{Id})^*(\mathscr{S}_X \times I)$. This completes the proof. □

Remark 5.4.3. If X is such that all self-homeomorphisms of X must act trivially on $H^3(X, \partial X; \mathbb{Z}/2)$, for example if $\pi_1(X)$ is cyclic, then Proposition 5.4.2 actually gives that the Casson-Sullivan invariant defines a group homomorphism from the pseudo-mapping class group. In such cases, one can combine this result with Corollary 5.3.4 to obtain that the Casson-Sullivan invariant defines a group homomorphism even if we start with a smoothable X, i.e. without picking a smooth structure.

Let X and X' be 4-dimensional smooth manifolds and let $f\colon X \to X'$ be an orientation-preserving homeomorphism restricting to a fixed diffeomorphism on ∂X. Then, up to isotopy, we may assume by isotopy extension [EK71], uniqueness of normal bundles [FQ90, Chapter 9.3], and the calculation of the mapping class group of S^3 [Cer68], that f restricts to the identity map on some disc. Hence we get a well-defined homeomorphism

$$f_\#\colon X\#(S^2\times S^2)\to X'\#(S^2\times S^2)$$

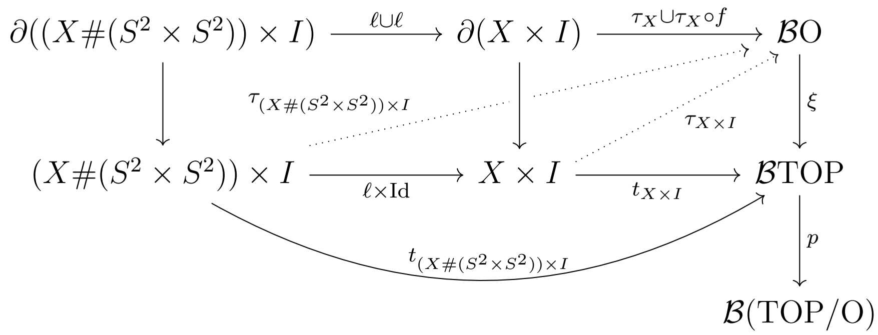
by extending f onto the $S^2\times S^2$ summand via the identity map.

Proposition 5.4.4. *The standard degree one map that collapses the $S^2\times S^2$ connected-summand $\ell\colon X\#(S^2\times S^2)\to X$ induces an isomorphism*

$$\ell^*\colon H^3(X,\partial X;\mathbb{Z}/2)\to H^3(X\#(S^2\times S^2),\partial X;\mathbb{Z}/2)$$

such that $\ell^*(\mathrm{cs}(f))=\mathrm{cs}(f_\#)$.

Proof. Let $\mathscr{S}_X$ denote the smooth structure on X and let $\mathscr{S}_{X\#S^2\times S^2}$ denote the induced smooth structure on $X\#(S^2\times S^2)$. Consider the following diagram.



Since $f_\#$ restricts to the identity map on the $S^2\times S^2$ summand and the tangent bundle of $S^2\times S^2$ is stably trivial, the stable tangent microbundle $t_{(X\#(S^2\times S^2))\times I}$ is homotopic to the stable tangent microbundle $t_{X\times I}$ precomposed with the map $\ell\times\mathrm{Id}$. This means that $\varpi\,\mathrm{cs}(f_\#)$, the homotopy class of $p\circ t_{(X\#(S^2\times S^2))\times I}$ extended by the null-homotopies given by $\mathscr{S}_{X\#(S^2\times S^2)}$ and $f_\#^*(\mathscr{S}_{X'\#(S^2\times S^2)})$ is relatively homotopic to the homotopy class of $p\circ t_{X\times I}\circ(\ell\times\mathrm{Id})$ extended by the null-homotopies given by precomposing the null-homotopies corresponding to $\mathscr{S}_X$ and $f^*\mathscr{S}_{X'}$ with the map ℓ. Since $(\ell\times\mathrm{Id})*(\varpi\,\mathrm{cs}(f))=\varpi(\ell^*\,\mathrm{cs}(f))$, it follows that $\ell^*(\mathrm{cs}(f))=\mathrm{cs}(f_\#)$. □

Proposition 5.4.5. *Let X and X' be a pair of smooth 4-manifolds and let $f\colon X\to X'$ be a homeomorphism restricting to a fixed diffeomorphism on ∂X. If $\mathrm{cs}(f)=0$ then there exists a non-negative integer k such that $f_\#\colon X\#(\#^kS^2\times S^2)\to X'\#(\#^kS^2\times S^2)$ is pseudo-isotopic to a diffeomorphism.*

This result is stated and proved in [FQ90, Section 8.6]. However, the proof is somewhat dispersed in the book and many of the details are not given. For this reason, we give the full proof below.

Proof of Proposition 5.4.5. Let $W := W(f)$ be the mapping cylinder of f. Since $\mathrm{cs}(f) = 0$, then by definition we can smooth the topological stable normal bundle of W relative to the given smoothings of the stable tangent bundles of the boundary. By Kirby-Siebenmann [KS77, Essay IV, Theorem 10.1], we can realise this by a smooth structure on W extending the given structures on the boundary. Note that here we have crucially used that the dimension of W is at least five and at most seven. This allows us to view W as a (relative) smooth h-cobordism (W, X, X'). Note that W is topologically a product (by definition) but that it is not necessarily a product smoothly, and that W already restricts to a smooth product on ∂X. We aim to turn W into a smooth product cobordism via stabilisations.

We may assume in the standard way that W has only 2- and 3-handles in its relative handle decomposition (for a reference, see [BKK+21, §20.1]). Since W is topologically a product we know that the 2- and 3-handles algebraically cancel. Let $\mathcal{W}$ be the collection of immersed Whitney discs in $X \times \{1/2\}$ for the pairs of cancelling intersections of the descending manifolds for the 3-handles and the ascending manifolds for the 2-handles. If these discs were embedded disjointly then we could use Whitney moves on the discs to force the 2- and 3-handles to geometrically cancel and hence W could be made into a smooth product cobordism. Let $p \in D_1 \cap D_2$ be an intersection point for two Whitney discs $D_1, D_2 \in \mathcal{W}$ (potentially $D_1 = D_2$) and let α be an arc in W from $X \times \{0\}$ to $X \times \{1\}$ which intersects D_1 and D_2 exactly once at p and is disjoint from all other discs in $\mathcal{W}$. Let $q = (q_1, q_2)$ be a point in $S^2 \times S^2$. Then we form a new cobordism

$$W' := \Big(W \setminus \nu\alpha\Big) \cup_{\partial\nu\alpha = (\partial\nu q)\times I} \Big((S^2 \times S^2 \setminus \nu q) \times I\Big).$$

Let $\mathcal{W}'$ be the set of Whitney discs for the pairs of cancelling intersections of the 2- and 3- handles of W'. Note that $\mathcal{W}'$ is the same as $\mathcal{W}$ except we can replace D_1 and D_2 with D_1' and D_2', respectively, defined as

$$\begin{aligned} D_1' :=& \Big(D_1 \cap (W \setminus \nu\alpha)\Big) \cup \Big((S^2 \times \{q_2\}) \cap (S^2 \times S^2 \setminus \nu q)\Big), \\ D_2' :=& \Big(D_2 \cap (W \setminus \nu\alpha)\Big) \cup \Big((\{q_1\} \times S^2) \cap (S^2 \times S^2 \setminus \nu q)\Big). \end{aligned}$$

The number of intersections between D_1' and D_2' is one fewer than the number of intersections between D_1 and D_2; we have effectively removed an intersection point. This is known as the Norman trick. Repeating this procedure for all intersections between Whitney discs, eventually we produce a smooth cobordism W'' which is topologically a product $W'' \approx X\#(\#^k S^2 \times S^2) \times I$ for some non-negative integer k. The set of Whitney discs for the pairs of cancelling intersections of the 2- and 3-handles $\mathcal{W}''$ consists now of disjointly embedded Whitney discs, and hence (as noted above) we may assume that W'' is smoothly a product cobordism.

Let $X_\# := X\#(\#^k S^2 \times S^2)$, $X'_\# := X'\#(\#^k S^2 \times S^2)$ and let $\mathscr{S}$, $\mathscr{S}'$ denote the smooth structures on $X_\#$ and $X'_\#$, respectively. Now W'' is the mapping cylinder of $f_\#$, i.e. $W'' \cong X_\# \times I$, with the smooth structure $\mathscr{S}_{W''}$ where $\mathscr{S}_{W''}|_{X_\# \times \{0\}} = \mathscr{S}$

and $\mathscr{S}_{W''}|_{X_\# \times \{1\}} = f_\#^*(\mathscr{S}')$. In the previous paragraph, we concluded that W'' was diffeomorphic to a product. In other words, we have a homeomorphism

$$F \colon X_\# \times I \to X_\# \times I$$

such that $F^*(\mathscr{S}' \times I) = \mathscr{S}''_W$. In the language of Section 2.3 this means we have produced a pseudo-isotopy between the smooth structures $\mathscr{S}$ and $f_\#^*(\mathscr{S}')$ and, by the same argument as in the third paragraph of the proof of Proposition 2.3.2, this means that $f_\#$ is pseudo-isotopic to a diffeomorphism. □

To summarise, Proposition 5.4.4 and Proposition 5.4.5 together tell us that the Casson-Sullivan invariant is the stable obstruction to pseudo-smoothing homeomorphisms of 4-manifolds, much as the Kirby-Siebenmann invariant is the stable obstruction to smoothing 4-manifolds.

§ 5.4.1 | Non-compact 4-manifolds

Non-compact 4-manifolds have the property that they are easier to smooth than their compact counterparts, in the sense that non-compact 4-manifolds always admit smooth structures (and hence compact 4-manifolds can always be smoothed away from a point) [FQ90, Chapter 8.2]. In light of this, one might wonder whether a stronger result than Proposition 5.4.5 holds if we assume our manifold is non-compact. We briefly explain what happens in this case. It will be helpful to recall the definition of concordance and sliced concordance of smooth structures (see Definition 2.1.10).

Proposition 5.4.6. *Let X be a topological 4-manifold with $\mathscr{S}$ and $\mathscr{S}'$ smooth structures on X, and let $f \colon X \to X$ be a homeomorphism restricting to a fixed diffeomorphism on ∂X with respect to the smooth structures $\mathscr{S}|_{\partial X}$ and $\mathscr{S}'|_{\partial X}$. If $\mathrm{cs}(f) = 0$, then $\mathscr{S}$ and $f^*(\mathscr{S}')$ are concordant. If X is non-compact and $\mathrm{cs}(f) = 0$, then $\mathscr{S}$ and $f^*(\mathscr{S}')$ are sliced concordant.*

This proposition essentially follows (in the non-compact case) from the following theorem.

Theorem 5.4.7 ([Sie71, Theorem 4.4],[FQ90, Theorem 8.7B])**.** *There is a one-to-one correspondence between sliced concordance classes of smooth structures on a non-compact 4-manifold with homotopy classes of liftings of the stable tangent microbundle to $\mathcal{B}\mathrm{O}$.*

Remark 5.4.8. Note that although in [FQ90] they state the above theorem for concordance classes rather than sliced concordance classes, the references they refer to for the proof give it for sliced concordance classes.

Proof of Proposition 5.4.6. The fact that $\mathscr{S}$ and $f^*(\mathscr{S}')$ are concordant is clear in the context of the proof of Proposition 5.4.5. In fact, this is exactly what the the first paragraph of the proof establishes. If $\mathrm{cs}(f) = 0$, then by Lemma 5.3.1 and Proposition 5.3.3 the lifts of the stable tangent microbundle corresponding to $\mathscr{S}$ and $f^*(\mathscr{S}')$ are homotopic, so by Theorem 5.4.7 these smooth structures are sliced concordant. □

Remark 5.4.9. Note that an important part of the proof of Theorem 5.4.7 is that the map

$$\mathrm{TOP}(4)/\mathrm{O}(4) \to \mathrm{TOP}/\mathrm{O}$$

is 5-connected [FQ90, Theorem 9.7A], and this theorem rests on the result of Quinn [Qui86] that $\pi_4(\mathrm{TOP}(4)/\mathrm{O}(4)) = 0$, the proof of which was found to contain a gap. The proof, however, was recently corrected in [GGH$^+$23].

A concordance, sliced or otherwise, between the smooth structures $\mathscr{S}$ and $f^*(\mathscr{S}')$ does not obviously give any nice statement about the properties of f itself. Hence, the author does not know of an interpretation of Proposition 5.4.6 in terms of the smoothability of the homeomorphism (c.f. Section 2.3).

§ 5.5 | A connected-sum over a circle formula for the Casson-Sullivan invariant

This subsection is devoted to proving a connected-sum along a circle formula for the Casson-Sullivan invariant. We shall start by giving the necessary definitions.

Definition 5.5.1. Let X_1 and X_2 be a pair of smooth 4-manifolds and let $\gamma_i \subset X_i$ be a pair of framed, embedded circles. Then we define the *connected-sum over* γ_1, γ_2 to be the smooth manifold

$$X_1 \#_{\gamma_1 = \gamma_2} X_2 := (X_1 \setminus \nu\gamma_1) \cup_\varphi (X_2 \setminus \nu\gamma_2)$$

where the gluing is performed using the orientation reversing map $\varphi\colon S^1 \times S^2 \to S^1 \times S^2$ which sends $(x, y) \to (x, a(y))$ for $a\colon S^2 \to S^2$ the antipodal map. For a precise description of how this gives a well-defined smooth manifold, see [Kos93, §VI.4].

Let X_1, X_2, X_1' and X_2' be two pairs of smooth, orientable, compact 4-manifolds and let $\gamma_i \subset X_i$ and $\gamma_i' \subset X_i'$ be two pairs of embedded circles for $i = 1, 2$. Since our manifolds are orientable, these circles admit framings which we will use implicitly (the choice of framings will not be important). Furthermore, let $f_i\colon X_i \to X_i'$ be a pair of homeomorphisms such that $(f_i)_*[\gamma_i] = [\gamma_i'] \in \pi_1(X_i')$. Then, up to isotopy, we may assume by isotopy extension ([EK71]), uniqueness of normal bundles ([FQ90, Chapter 9.3]), and the calculation of the mapping class group of $S^1 \times S^2$ [Glu62] that the f_i are either the identity map on a tubular neighbourhood of the curve γ_i or they are the *Gluck twist* map

$$\begin{aligned} T\colon S^1 \times D^3 &\to S^1 \times D^3 \\ (t, x) &\mapsto (t, R_t(x)) \end{aligned} \tag{5.5.1}$$

where R_t denotes the (positive) rotation map of D^3 around the (oriented) straight line from the south pole to the north pole by an angle of t (we have used the identification $S^1 \cong [0, 2\pi]/0 \sim 2\pi$). If the homeomorphisms f_i are both of the same type as above (i.e. both are identity maps or both are twist maps on tubular neighbourhoods of γ_i) we may then define the connected-sum of these homeomorphisms over the curves γ_i to be

$$f_\# \colon X_1 \#_{\gamma_1 = \gamma_2} X_2 \to X_1' \#_{\gamma_1' = \gamma_2'} X_2'$$

as f_i on $(X_i \setminus \nu\gamma_i)$.

Lemma 5.5.2. *We have the following isomorphism of groups.*

$$H^3(X_1 \#_{\gamma_1=\gamma_2} X_2, \partial; \mathbb{Z}/2) \cong \frac{H^3(X_1, \partial X_1; \mathbb{Z}/2) \oplus H^3(X_2, \partial X_2; \mathbb{Z}/2)}{\mathrm{PD}^{-1}[\gamma_1] \sim \mathrm{PD}^{-1}[\gamma_2]}$$

Proof. Let $X_\# := X_1 \#_{\gamma_1=\gamma_2} X_2$. By considering the long exact sequence for the triple

$$(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1, \partial X_\#)$$

and the Mayer-Vietoris sequence for the decomposition

$$(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1) = ((X_1 \setminus \nu\gamma_1) \cup (X_2 \setminus \nu\gamma_2), \partial(X_1 \setminus \nu\gamma_1) \cup \partial(X_2 \setminus \nu\gamma_2)),$$

we obtain the following commutative diagram (with $\mathbb{Z}/2$-coefficients suppressed).

$$\begin{array}{ccccccc}
 & & H^2(\partial\overline{\nu}\gamma_1) & \xrightarrow{\quad\Delta\quad} & H^2(\partial\overline{\nu}\gamma_1) \oplus H^2(\partial\overline{\nu}\gamma_1) & & \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
0 & \longrightarrow & H^3(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1) & \xrightarrow{\cong} & H^3(X_1 \setminus \nu\gamma_1, \partial) \oplus H^3(X_2 \setminus \nu\gamma_2, \partial)) & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & H^3(X_\#, \partial) & & & & \\
 & & \downarrow{\scriptstyle 0} & & & & \\
 & & H^3(\partial\overline{\nu}\gamma_1) & \xrightarrow{\quad\Delta\quad} & H^3(\partial\overline{\nu}\gamma_1) \oplus H^3(\partial\overline{\nu}\gamma_1) & & \\
 & & \downarrow & & \downarrow{\scriptstyle\cong} \qquad \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & H^4(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1) & \xrightarrow{\cong} & H^4(X_1, \partial X_1) \oplus H^4(X_2, \partial X_2) & \longrightarrow & 0
\end{array}$$

Here the vertical maps on the left are from the previously stated long exact sequence of the triple, the horizontal isomorphism comes from the previously stated Mayer-Vietoris sequence, and Δ denotes the diagonal map. That the map marked as the zero map is such follows from seeing that the vertical maps in the lower square are injective.

Poincaré duality maps and the map induced by the inclusion $X_i \setminus \nu\gamma_i \subset X_i$ give isomorphisms

$$H^3(X_i \setminus \nu\gamma_1, \partial) \cong H_1(X_i) \cong H_1(X_i \setminus \nu\gamma_i) \cong H^3(X_i, \partial)$$

for $i = 1, 2$. Using this, we get an exact sequence

$$\mathbb{Z}/2 \cong H^2(\partial\overline{\nu}\gamma_1) \longrightarrow \begin{matrix} H^3(X_1, \partial X_1) \\ \oplus \\ H^3(X_2, \partial X_2) \end{matrix} \longrightarrow H^3(X_\#, \partial X_\#) \longrightarrow 0$$

and so to prove the lemma it suffices to understand the first map. From the definition one can see that this is the map which sends the generator of $H^2(\partial\overline{\nu}\gamma_1) \cong \mathbb{Z}/2$ to the element

$$(\mathrm{PD}^{-1}[\gamma_1], \mathrm{PD}^{-1}[\gamma_2]) \in H^3(X_1, \partial X_1) \oplus H^3(X_2, \partial X_2).$$

□

From Lemma 5.5.2 we have a map

$$Q\colon H^3(X_1, \partial X_1) \oplus H^3(X_2, \partial X_2) \to H^3(X_1\#_{\gamma_1=\gamma_2}X_2, \partial(X_1\#_{\gamma_1=\gamma_2}X_2)) \tag{5.5.2}$$

given by taking the quotient.

The rest of this subsection will be devoted to proving the following theorem.

Theorem 5.5.3. *Let X_1, X_2, X_1' and X_2' be two pairs of compact, connected, smooth, orientable 4-manifolds and let $\gamma_i \subset X_i$ and $\gamma_i' \subset X_i'$ be two pairs of embedded circles, with $f_i\colon X_i \to X_i'$ a pair of homeomorphisms such that $(f_i)_*[\gamma_i] = [\gamma_i'] \in \pi_1(X_i')$, and such that the connected-sum homeomorphism*

$$f_\# := f_1\#_{\gamma_1=\gamma_2}f_2\colon X_1\#_{\gamma_1=\gamma_2}X_2 \to X_1'\#_{\gamma_1'=\gamma_2'}X_2'$$

is defined. Let Q be the map in Equation (5.5.2). Then

$$\mathrm{cs}(f_\#) = Q(\mathrm{cs}(f_1), \mathrm{cs}(f_2)).$$

The proof of this theorem relies on a sequence of diagram chasing, using the relevant Mayer-Vietoris exact sequences and long exact sequences of the triple. Although we have already used these, the setup here will be more complicated and so we recall them. The long exact sequence for the triple of CW-complexes $W \supset B \supset A$ in cohomology (with $\mathbb{Z}/2$-coefficients suppressed) is

$$\cdots \to H^k(W, B) \to H^k(W, A) \to H^k(B, A) \to H^{k+1}(W, B) \to \cdots \tag{5.5.3}$$

and the fully relative Mayer-Vietoris sequence for the pair of CW-complexes $(W, Y) = (A \cup B, C \cup D)$ (with $\mathbb{Z}/2$-coefficients suppressed) is

$$\cdots \to H^k(W, Y) \to \begin{matrix} H^k(A, C) \\ \oplus \\ H^k(B, D) \end{matrix} \to H^k(A \cap B, C \cap D) \to H^{k+1}(W, Y) \to \cdots \tag{5.5.4}$$

See [Hat02, p.200/204] for these standard exact sequences.

First we give the notation for the setup.

Setup 5.5.4. Let X_1, X_2, X_1' and X_2' be two pairs of compact, connected, smooth, orientable 4-manifolds, let $\gamma_i \subset X_i$ be a pair of embedded circles, and let $f_i\colon X_i \to X_i'$ be a pair of homeomorphisms such that $(f_i)_*[\gamma_i] = [\gamma_i'] \in \pi_1(X_i')$ and such that the f_i restrict to fixed diffeomorphisms $(f_i)_0\colon \partial X_i \to \partial X_i'$. Denote by $X_\#$ and $X_\#'$ the connected-sum $X_1 \#_{\gamma_1=\gamma_2} X_2$ and $X_1' \#_{\gamma_1=\gamma_2} X_2'$, respectively, and by $f_\#\colon X_\# \to X_\#'$ the connected-sum homeomorphism (which we assume to be defined). We then set up the following notation:

(i) Let $W := M_{f_\#} = \frac{(X_\# \times I) \cup_\partial X_\#'}{(x,1) \sim f_\#(x)}$.

(ii) Let $E := \partial(\nu\gamma_1) \times I \subset W$.

(iii) Let $F := \partial W = (X_\# \times \{0\}) \cup (\partial X_\# \times I) \cup (X_\# \times \{1\}) \subset W$.

(iv) Let $A := (X_1 \setminus \partial(\nu\gamma_1)) \times I \subset W$ and let $B := (X_2 \setminus \partial(\nu\gamma_2)) \times I \subset W$.

(v) Let $F_A := F \cap A$ and let $F_B := F \cap B$.

(vi) Let $C := F_A \cup E \subset A$ and let $D := F_B \cup E \subset B$.

(vii) Let $Y := C \cup D$.

Note that $W = A \cup B$ and $A \cap B = C \cap D = E$. See Figure 5.1.

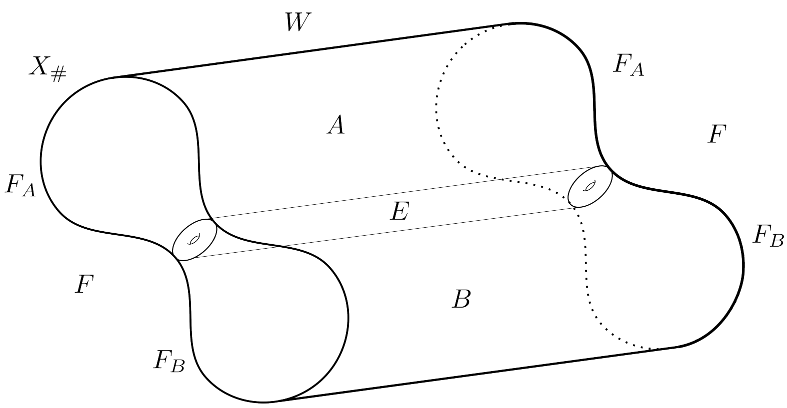


Figure 5.1: A dimension-reduced picture of Setup 5.5.4 where the pictured tori denote the connected-sum $S^1 \times S^2$. Note that the labels for C and D have been omitted.

Combining the sequences (5.5.3) and (5.5.4) for the triples (W, Y, F), (A, C, F_A), and (B, D, F_B) and the pairs $(W, Y) = (A \cup B, C \cup D)$, $(W, F) = (A \cup B, F_A \cup F_B)$, and $(Y, F) = (C \cup D, F_A \cup F_B)$ we obtain the following commutative diagram which we will use extensively. In what follows, for all of our cohomology groups we are using $\mathbb{Z}/2$-coefficients but we will suppress these in the diagrams.

$$\begin{array}{ccccccc}
 & & H^3(Y,F) & \longrightarrow & H^3(C,F_A)\oplus H^3(D,F_B) & & \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
H^3(A\cap B, C\cap D) & \longrightarrow & H^4(W,Y) & \longrightarrow & H^4(A,C)\oplus H^4(B,D) & \longrightarrow & H^4(A\cap B, C\cap D) \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
H^3(A\cap B, F_A\cap F_B) & \longrightarrow & H^4(W,F) & \longrightarrow & H^4(A,F_A)\oplus H^4(B,F_B) & \longrightarrow & H^4(A\cap B, F_A\cap F_B) \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
 & & H^4(Y,F) & & H^4(C,F_A)\oplus H^4(D,F_B) & &
\end{array}$$

It will be useful to simplify this diagram. First, note that the leftmost and rightmost groups on the second line vanish because $A\cap B = C\cap D$ (further observe that this means the remaining non-trivial horizontal map on the second line must be an isomorphism). The topmost groups and the leftmost group on the third line are all isomorphic to $H^3(E,\partial E)\cong \mathbb{Z}/2$. We can replace all of the remaining outer groups with zeroes since these maps must be zero maps (one can explicitly see this by continuing the sequences and using commutativity along with Poincaré duality). To illustrate this, we will show that the bottom vertical maps are zero. Showing that the map to $H^4(A\cap B, F_A\cap F_B)$ is zero is analogous. We start by continuing the sequences at the bottom of the diagram to obtain the following diagram.

$$\begin{array}{ccccccc}
 & & H^4(W,F) & \longrightarrow & H^4(A,F_A)\oplus H^4(B,F_B) & & \\
 & & \downarrow a & & \downarrow b & & \\
H^3(C\cap D, F_A\cap F_B) & \xrightarrow{f} & H^4(Y,F) & \xrightarrow{c} & H^4(C,F_A)\oplus H^4(D,F_B) & & \\
 & & \downarrow d & & \downarrow e & & \\
0 & \longrightarrow & H^5(W,Y) & \xrightarrow{\cong} & H^5(A,C)\oplus H^5(B,D) & \longrightarrow & 0
\end{array}$$

We want to show that the maps a and b are both the zero maps. It suffices to show that d and e are injections. By Poincaré duality, e is dual to direct sum of inclusion induced maps

$$H_0(E)\oplus H_0(E)\to H_0(A)\oplus H_0(B)$$

which is clearly an isomorphism since both A, B and E are connected. Hence e is injective. Now we claim that f is the zero map, since the previous map in the Mayer-Vietoris sequence is the diagonal map and hence is injective. This means that c is injective, and so by commutativity d must be injective also, and hence a and b are both the zero maps. The preceding simplification yields the following diagram.

$$\begin{array}{ccccccc}
 & & \mathbb{Z}/2 & \xrightarrow{\quad\Delta\quad} & \mathbb{Z}/2 \quad \oplus \quad \mathbb{Z}/2 & & \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle i_A} \qquad \downarrow{\scriptstyle i_B} & & \\
0 & \longrightarrow & H^4(W,Y) & \xrightarrow{\cong} & H^4(A,C)\oplus H^4(B,D) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j_A} \qquad {\scriptstyle j_B}\downarrow & & \\
\mathbb{Z}/2 & \longrightarrow & H^4(W,F) & \longrightarrow & H^4(A,F_A)\oplus H^4(B,F_B) & \longrightarrow & 0\\
 & & \downarrow & & \searrow \quad \swarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{5.5.5}$$

We note that the topmost horizontal map is the diagonal map $\Delta\colon x \mapsto (x,x)$.

Proof of Theorem 5.5.3. We will use the notation from Setup 5.5.4 throughout. Consider the pair $(W,Y) = (A \cup B, C \cup D)$. First, we will consider a special case, namely when $X_2 = S^1 \times S^3$, $\gamma_2 = S^1 \times \{\mathrm{pt}\}$ and $f_2 = \mathrm{Id}_{X_2}$. We will then prove the general case of the theorem via our consideration of the special case.

Let $X_2 = S^1 \times S^3 = X_2'$, let $\gamma_2 = S^1 \times \{\mathrm{pt}\}$ and let $f_2 = \mathrm{Id}_{X_2}$. Then $X_2 \backslash (\nu\gamma_2) \cong S^1 \times D^3$, so hence $X_\# \cong X_1$ and $f_\# = f_1$. The map i_B is then Poincaré dual to the inclusion induced map $H_1(S^1 \times S^2 \times I) \to H_1(S^1 \times D^3 \times I)$ and hence is injective. It follows by commutativity that r is injective. Consider any element $x \in H^4(W,F)$. From the diagram and injectivity of r, we can see that there are two possible lifts of x in $H^4(W,Y)$ that differ by $r(z)$ where $\mathbb{Z}/2 = \mathbb{Z}/2\langle z\rangle$. We will define a preferred lift $t(x)$ by specifying that the element $t(x) = (t_1(x), 0) \in H^4(A,C) \oplus H^4(B,D)$, which uniquely determines t. Now consider $\varpi\,\mathrm{cs}(f_1) \in H^4(W,F)$. By naturality of the Kirby-Siebenmann invariant (Remark 5.2.8), q maps $\mathrm{ks}(W,Y) = (\mathrm{ks}(A,C), \mathrm{ks}(B,D)) \in H^4(W,Y)$ to $\varpi\,\mathrm{cs}(f_1)$, and, by the definition of f_2, we have that $\mathrm{ks}(B,D) = 0$. This means that $t(\varpi\,\mathrm{cs}(f_1)) = \mathrm{ks}(W,Y)$.

Now consider general $X_1, X_2, X_1', X_2', f_1, f_2$. What we have described above is a map

$$a\colon H^3(X_1, \partial X_1) \to H^4(A,C)$$

which sends $\varpi\,\mathrm{cs}(f_1)$ to an element $\alpha := a(\varpi\,\mathrm{cs}(f_1))$ for any homeomorphism $f_1\colon X_1 \to X_1$. Similarly, by considering the reverse special case where $X_1 = S^1 \times S^3 = X_1'$, $f_1 = \mathrm{Id}_{X_1}$ and where f_2 is any homeomorphism $f_2\colon X_2 \to X_2'$, we obtain a map $b\colon H^3(X_2, \partial X_2) \to H^4(B,D)$ and consequently an element $\beta := b(\varpi\,\mathrm{cs}(f_2)) \in H^4(B,D)$. This produces an element

$$(\alpha,\beta) \in H^4(A,C) \oplus H^4(B,D) \cong H^4(W,Y)$$

and then we map this down using q to an element $q(\alpha,\beta)$. We have that $q(\alpha,\beta) = \varpi\,\mathrm{cs}(f_\#)$, as by construction $(\alpha,\beta) = (\mathrm{ks}(A,C), \mathrm{ks}(B,D))$ and by naturality of the Kirby-Siebenmann invariant, this must map to $\mathrm{ks}(W,F) = \varpi\,\mathrm{cs}(f_\#)$ via q, since q is inclusion induced.

So we have constructed a map

$$P\colon H^3(X_1, \partial X_1) \oplus H^3(X_2, \partial X_2) \to H^3(X_1 \#_{\gamma_1=\gamma_2} X_2, \partial(X_1 \#_{\gamma_1=\gamma_2} X_2))$$

which sends $(\mathrm{cs}(f_1), \mathrm{cs}(f_2)) \mapsto \mathrm{cs}(f_\#)$.

It remains to show that this map is equal to the map Q (see 5.5.2). We show this now. First, analogously to in Setup 5.5.4, consider the long exact sequences for the triples

(i) $(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1, \partial X_\#)$,

(ii) $(X_i \setminus \nu\gamma_i, \partial(X_i \setminus \nu\gamma_i), \partial X_i)$,

and the relative Mayer-Vietoris sequences for the pairs

(i) $(X_\#, \partial X_\#) = ((X_1 \setminus \nu\gamma_1) \cup (X_2 \setminus \nu\gamma_2), \partial X_1 \cup \partial X_2)$

(ii) $(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1) = ((X_1 \setminus \nu\gamma_1) \cup (X_2 \setminus \nu\gamma_2), \partial(X_1 \setminus \nu\gamma_1) \cup \partial(X_2 \setminus \nu\gamma_2))$.

These sequences give the following commutative diagram analogous to (5.5.5) (note that we have simplified the notation by writing ∂ on its own to refer to the boundary of a manifold when it is clear by context which manifold is being referred to).

$$\begin{array}{ccccccc}
 & & \mathbb{Z}/2 & \xrightarrow{\quad\Delta\quad} & \mathbb{Z}/2 \;\oplus\; \mathbb{Z}/2 & & \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
0 & \longrightarrow & H^3(X_\#, \partial X_\# \sqcup \partial\overline{\nu}\gamma_1) & \xrightarrow{\cong} & H^3(X_1 \setminus \nu\gamma_1, \partial) \oplus H^3(X_2 \setminus \nu\gamma_2, \partial)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \qquad \downarrow & & \\
\mathbb{Z}/2 & \longrightarrow & H^3(X_\#, \partial) & \longrightarrow & H^3(X_1 \setminus \nu\gamma_1, \partial X_1) \oplus H^3(X_2 \setminus \nu\gamma_2, \partial X_2) & \longrightarrow & 0 \\
 & & \downarrow & & \searrow \quad \swarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{5.5.6}$$

One can see that this diagram (5.5.6) is isomorphic to (5.5.5) using the isomorphism given in Definition 5.2.9 (or the analogous isomorphisms). Since those isomorphisms come from Poincaré duality and inclusion maps, the two parallel diagrams must commute. This allows us to reinterpret the construction of our map P on the level of the 4-manifolds themselves, rather than the mapping cylinders, which then allows us to relate this map to Q.

Consider the following diagram which, aside from the lowermost map and group, splits as a direct sum of exact sequences.

$$
\begin{array}{ccccc}
\begin{matrix} H^3(X_1, \partial X_1 \sqcup \partial \overline{\nu}\gamma_1) \\ \oplus \\ H^3(X_2, \partial X_2 \sqcup \partial \overline{\nu}\gamma_2) \end{matrix} & \twoheadrightarrow & H^3(X_1, \partial) \oplus H^3(X_2, \partial) & \xrightarrow[\cong]{\text{PD}} & \begin{matrix} H_1(X_1) \\ \oplus \\ H_1(X_2) \end{matrix} \\
\downarrow \cong & \overset{t}{\swarrow} & & \overset{t'}{\searrow} & \uparrow \\
\begin{matrix} H^3(X_1 \setminus \nu\gamma_1, \partial) \oplus H^3(\overline{\nu}\gamma_1, \partial) \\ \oplus \\ H^3(X_2 \setminus \nu\gamma_2, \partial) \oplus H^3(\overline{\nu}\gamma_2, \partial) \end{matrix} & & \xrightarrow[\cong]{((\text{PD},\text{PD}),(\text{PD},\text{PD}))} & & \begin{matrix} H_1(X_1 \setminus \nu\gamma_1) \oplus H_1(\overline{\nu}\gamma_1) \\ \oplus \\ H_1(X_2 \setminus \nu\gamma_2) \oplus H_1(\overline{\nu}\gamma_2) \end{matrix} \\
& \underset{(\text{pr}_1, \text{pr}_1)}{\searrow} & & \underset{(\text{PD}^{-1} \circ \text{pr}_1, \text{PD}^{-1} \circ \text{pr}_1)}{\swarrow} & \\
& & H^3(X_1 \setminus \nu\gamma_1, \partial) \oplus H^3(X_2 \setminus \nu\gamma_2, \partial) & & \\
& & \downarrow & & \\
& & H^3(X_\#, \partial) & &
\end{array}
\tag{5.5.7}
$$

The two vertical maps are direct sums of the respective Mayer-Vietoris sequences. The top-left horizontal map comes from the direct sum of the respective long exact sequences of triples. Ignoring the notated lifts t and t' for now, we show that this diagram commutes. The commuting of the lower triangle is trivial. We now show that the top rectangle commutes. Since the rectangle is a direct sum of two diagrams, it suffices to show that one of the diagram summands commutes. We write this below, where we will drop the indices (i.e. $X_1 = X$, etc.).

$$
\begin{array}{ccc}
H^3(X, \partial) & \xrightarrow[\cong]{\text{PD}} & H_1(X) \\
\uparrow & & \uparrow \\
H^3(X, \partial X \sqcup \partial \overline{\nu}\gamma) & & \\
\downarrow \cong & & \\
H^3(X \setminus \nu\gamma, \partial) \oplus H^3(\overline{\nu}\gamma, \partial) & \xrightarrow[\cong]{(\text{PD},\text{PD})} & H_1(X \setminus \nu\gamma) \oplus H_1(\overline{\nu}\gamma)
\end{array}
$$

Let $x \in H^3(X, \partial X \sqcup \partial \overline{\nu}\gamma)$ and let y denote its image in $H^3(X, \partial)$. By the vertical isomorphism, we see that x maps to an element $(\text{PD}^{-1}\, \alpha_1, \text{PD}^{-1}\, \alpha_2) \in H^3(X \setminus \nu, \partial) \oplus H^3(\overline{\nu}\gamma, \partial)$, where α_1 is the homology class of some curve in X and α_2 is trivial or is the homology class of γ. Mapping this pair to the right and then up gives the class $\alpha_1 + \alpha_2$, which is the homology class of some curve in X. We now need to show that y is Poincaré dual to $\alpha_1 + \alpha_2$. Since we are using $\mathbb{Z}/2$-coefficients, we will view these cohomology groups as the hom-duals of the respective homology groups. The element y is defined such that it evaluates on the relative cycle s in X first by considering s as a further relative cycle $s' \in H_3(X, \partial X \sqcup \partial \overline{\nu}\gamma)$ and then evaluating using x. By the isomorphism in the corresponding Mayer-Vietoris sequence in homology, any such cycle splits as a pair (s_1, s_2) of relative cycles in $H_3(X \setminus \nu\gamma, \partial)$ and $H_3(\overline{\nu}\gamma, \partial)$, respectively. Hence, evaluating $x(s')$ is the same as evaluating $\alpha_1(s_1) + \alpha_2(s_2)$, and so y is Poincaré dual to $\alpha_1 + \alpha_2$. This completes the proof that the diagram commutes.

Consider again diagram 5.5.7 and let $(x_1, x_2) \in H^3(X_1, \partial) \oplus H^3(X_2, \partial)$. This pair is mapped by P by sending it down the left of the diagram, first by using the preferred lift t, and then using the given maps. The lift t is such that $t(x_1, x_2)$ is of the form $((t(x_1), 0), (t(x_2), 0))$. Reversely, the pair (x_1, x_2) is mapped by Q by sending it down the right of the diagram, first by using the preferred lift t', where $t'(x_1, x_2)$ is of the form $((t'(x_1), 0), (t'(x_2), 0))$, which is naturally given by first taking Poincaré duality and then the inverse of the map induced by the inclusion map (and then including into the direct sum). Then $t'(x_1, x_2)$ is mapped down again twice to $H^3(X_\#, \partial)$.

To finish the proof, we need to know that the middle triangle formed by the two lifts, t and t', commutes. The fact that the diagram commutes shows that

$$((\mathrm{PD}, \mathrm{PD}), (\mathrm{PD}, \mathrm{PD}))\,(t(x_1, x_2))$$

is a lift of $(\mathrm{PD}, \mathrm{PD})(x_1, x_2)$, and the definition of the lifts t and t' shows that it is the same lift as determined by $t(x_1, x_2)$, and hence the triangle commutes. □

§ 5.6 | A connected-sum formula for the Casson-Sullivan invariant

Before finishing this section we will note the following. Similarly to Theorem 5.5.3, we also have a formula for the Casson-Sullivan invariant under the actual connected-sum operation. Since all of the arguments are the same or simpler than for connected-summing over a circle, we will not give many details. Let X_1, X_2, X_1' and X_2' be two pairs of compact, connected topological 4-manifolds and let $f_i\colon X_i \to X_i'$ be a pair of homeomorphisms. In exactly the same way as for connected-summing over a circle (in fact it is simpler since we do not have to worry about embeddings of circles), we can form a connected-sum homeomorphism $f_\#\colon X_1 \# X_2 \to X_1' \# X_2'$. If we let $q_i\colon X_1 \# X_2 \to X_i$ denote the collapse maps onto the ith connected-summand, then we have the following formula.

Theorem 5.6.1. *Let X_1, X_2, X_1' and X_2' be two pairs of compact, connected, smooth, orientable 4-manifolds with $f_i\colon X_i \to X_i'$ a pair of homeomorphisms such that the connected-sum homeomorphism*

$$f_\# := f_1 \# f_2 \colon X_1 \# X_2 \to X_1' \# X_2'$$

is defined. Let q_i be the pair of collapse maps defined above. Then

$$\mathrm{cs}(f_\#) = q_1^* \,\mathrm{cs}(f_1) + q_2^* \,\mathrm{cs}(f_2).$$

We will not give the proof here, as it is exactly the same method as for proving Theorem 5.5.3 but with simpler arguments. In particular, having the degree one collapse is very useful (such collapse maps do not always exist for connected-sums over circles). This result will only be used for proving Corollary 6.0.2 in Section 6.2.3.

Chapter 6

Stable realisation of the Casson-Sullivan invariant

In this chapter we will prove that the Casson-Sullivan invariant is stably realisable. In particular, we will prove Theorem 1.2.2 from the introduction. We restate this theorem now.

Theorem 6.0.1. *Let X and X' be compact, connected, smooth, orientable 4-manifolds such that $X \cong X_0\#(S^2 \times S^2)$ and $X' \cong X_0'\#(S^2 \times S^2)$ where $X_0 \approx X_0'$, and let $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$. Then there exists a homeomorphism $f\colon X \to X'$ with $\mathrm{cs}(f) = \eta$.*

The idea of the proof is to first show that we can realise the Casson-Sullivan invariant in a specific case, and then to use Theorem 5.5.3 to realise the invariant in all cases stably. We will construct a homeomorphism

$$\sigma\colon (S^1 \times S^3)\#(S^2 \times S^2) \to (S^1 \times S^3)\#(S^2 \times S^2)$$

with $\mathrm{cs}(\sigma) \in H^3((S^1 \times S^3)\#(S^2 \times S^2); \mathbb{Z}/2)$ the non-trivial element. This will be Proposition 6.2.1. For this, we will use the work of Ronnie Lee from an unpublished letter [Lee70] (see also Scharlemann and Akbulut [Sch76, Akb99]). The material necessary from [Lee70] was typed up by the author in [Gal24b] and we will present that information here.

As a corollary of our methods to prove Theorem 6.0.1, we will demonstrate that smoothability of a homeomorphism depends on the isotopy class of the smooth structure, as is expected by Section 5.3. We state this now.

Corollary 6.0.2. *Let $X = (S^1 \times S^3)\#(S^1 \times S^3)\#(S^2 \times S^2)$ with the standard smooth structure and let $g\colon X \to X$ be the diffeomorphism which swaps the two $S^1 \times S^3$ connected-summands and is the identity on the $S^2 \times S^2$ connected-summand. Then there exists a smooth structure $\mathscr{S}'$ on X, which is diffeomorphic to the standard smooth structure, but is such that g is not stably pseudo-smoothable with respect to $\mathscr{S}'$.*

§ 6.0.1 | Chapter outline

We begin with Section 6.1, where we present the material from Ronnie Lee's note. In Section 6.2, we prove the stated results of this chapter.

§ 6.1 | Ronnie Lee's generator for $L_5(\mathbb{Z}[\mathbb{Z}])$

As was stated previously, the material here was originally presented in a letter [Lee70] addressed to Martin Scharlemann from some time in the 1970s.[1]

We begin with the definition of Wall's surgery obstruction groups, the L-groups.

Definition 6.1.1. Let $n \in \mathbb{Z}$ and π a finitely-presented group. Then the quadratic *L-group* $L_n(\mathbb{Z}[\pi])$ is defined differently depending on the residue of n modulo 4.

Even case (if $n \equiv 0, 2 \pmod 4$): $L_n(\mathbb{Z}[\pi])$ is defined as the set of stable equivalence classes of $(-1)^{n/2}$-quadratic forms over stably free $\mathbb{Z}[\pi]$-modules.

Odd case (if $n \equiv 1, 3 \pmod 4$): $L_n(\mathbb{Z}[\pi])$ is defined as the set of stable equivalence classes of $(-1)^{(n-1)/2}$-quadratic formations over stably free $\mathbb{Z}[\pi]$-modules.

For a reference on (quadratic) forms and formations see [Ran81, §1.6]. There are also *simple* L-groups, denoted by $L_n^s(\mathbb{Z}[\pi])$. We will not give the definition here, but for a reference see [Lüc23, §9.10].

Remark 6.1.2. The notation used here for the L-groups matches up with the notation used for K-theory. In principle one could consider L-groups of arbitrary rings with involution, but for our purposes we will only consider the L-theory of group rings with the standard involution and trivial orientation character. One should be careful when reading other sources, as $L_n(\mathbb{Z}[\pi])$ is often written instead as $L_n(\pi)$. In particular, Lee's note [Lee70] uses the other convention i.e. refers to $L_5(\mathbb{Z}[\mathbb{Z}])$ instead as $L_5(\mathbb{Z})$.

§ 6.1.1 | Constructing any generator

The aim of this section is to give a 'nice' description for the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$, but first we simply need to construct any generator. Shaneson splitting [Sha69, Theorem 5.1] gives us that

$$L_5(\mathbb{Z}[\mathbb{Z}]) \cong L_5(\mathbb{Z}) \times L_4(\mathbb{Z}).$$

Wall [Wal70, Theorem 13A.1] computes that $L_5(\mathbb{Z}) = 0$ and similarly that $L_4(\mathbb{Z}) \cong L_0(\mathbb{Z}) \cong \mathbb{Z}$, hence $L_5(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$, generated by the image of the generator of $L_4(\mathbb{Z})$ in $L_5(\mathbb{Z}[\mathbb{Z}])$. The injection $L_4(\mathbb{Z}) \hookrightarrow L_5(\mathbb{Z}[\mathbb{Z}])$ is given by taking a surgery problem and multiplying it by S^1. We have an isomorphism $L_4(\mathbb{Z}) \cong \mathbb{Z}\langle\sigma/8\rangle$ where σ denotes the signature, and so it is generated by the standard surgery problem associated to the E_8 manifold. Hence, $L_5(\mathbb{Z}[\mathbb{Z}])$ is generated by the induced surgery problem on $E_8 \times S^1$. We will now be more precise.

Consider the manifold P formed by plumbing along the E_8-lattice. The created manifold P is a manifold with boundary ∂P a homology 3-sphere Y, and by Freedman [FQ90, Corollary 9.3C] Y also bounds a contractible manifold which we will denote by B. In a slight abuse of notation, we then define $E_8 := P \cup_Y B$. By construction, E_8

[1] I would like to thank Ian Hambleton for providing me with a copy of this letter.

has intersection form given by the E_8-lattice

$$\lambda_{E_8} = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & 1 & & 1 & 2 \end{bmatrix} \tag{6.1.1}$$

and one can diagonalise this matrix to read off its signature (the computation can be found in full detail in [Sco05, Section 3.2]). This gives $\sigma(E_8) = 8$. This means that by Rokhlin's theorem [Rok52] this manifold is non-smoothable. By Wall, this means that the surgery problem corresponding to the standard degree-1 normal map

$$\varphi' \colon E_8 \to S^4$$

has surgery obstruction the generator of $L_4(\mathbb{Z})$ and hence the surgery problem corresponding to the map

$$\varphi := \varphi' \times \mathrm{Id}_{S^1} \colon E_8 \times S^1 \to S^4 \times S^1$$

has surgery obstruction the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$.

Now that we have constructed a surgery problem with the required generator as its obstruction, the aim is to find an algebraic description for this element in $L_5(\mathbb{Z}[\mathbb{Z}])$. For this, we follow the method given in [Wal70, §6]. It is not hard to see that our map φ is already 2-connected, since it is clearly an isomorphism on π_1, and $\pi_2(S^4 \times S^1) = 0$. The surgery kernel is therefore $\pi_3(\varphi) \cong H_2(E_8 \times S^1; \Lambda)$, where $\Lambda := \mathbb{Z}[\pi_1(E_8 \times S^1)] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[T^1, T^{-1}]$. Choose a Λ-basis for this, represented by eight disjoint embeddings $h_i \colon S^2 \times D^3 \hookrightarrow E_8 \times S^1$ corresponding to the basis given for the intersection form in (6.1.1), and let U denote the union of all of these embeddings $U := \cup_i h_i(S^2 \times D^3)$.

We have now split our manifold $E_8 \times S^1$ into two pieces: $\overline{E_8 \times S^1 \setminus U}$ and U. Furthermore, after cellular approximation we can assume that our map φ takes the form of a map of triads

$$\varphi : (E_8 \times S^1; (\overline{E_8 \times S^1 \setminus U}), U) \to (S^4 \times S^1; (\overline{S^4 \times S^1 \setminus D^5}), D^5).$$

An element of $L_5(\mathbb{Z}[\mathbb{Z}])$ is a formation. Every formation is equivalent to a formation of the form $(H_\varepsilon(F); F, G)$, but we can see this explicitly in our case. Since $\partial U \cong \sqcup_i S^2 \times S^2$, a disjoint union of embedded copies of $S^2 \times S^2$, $H_2(\partial U; \Lambda)$ is already the standard hyperbolic form over Λ with sixteen generators, where e_i corresponds to the ith copy of $S^2 \times \{\mathrm{pt}\}$ and f_i corresponds to the ith copy of $\{\mathrm{pt}\} \times S^2$.

More specifically, the formation corresponding to this surgery problem is given by $(H; F, G)$ where

$$\begin{aligned}
H &:= \ker\Big((\varphi\mid_{\partial U})_*\colon H_2(\partial U;\Lambda) \to H_2(\partial D^5;\varphi_*\Lambda)\Big),\\
F &:= \ker\Big((\varphi\mid_{U})_*\colon H_3(U,\partial U;\Lambda) \to H_3(D^5,\partial D^5;\varphi_*\Lambda)\Big),\\
G &:= \ker\Big((\varphi\mid_{\overline{E_8\times S^1\setminus U}})_*\colon H_3(\overline{E_8\times S^1\setminus U},\partial U;\Lambda) \to H_3(\overline{S^4\times S^1\setminus D^5},\partial D^5;\varphi_*\Lambda)\Big).
\end{aligned}$$

All of the restriction maps that we take above are the zero maps on their respective homology groups because their targets vanish, hence $H \cong H_2(\partial U;\Lambda)$, $F \cong H_3(U,\partial U;\Lambda)$ and $G \cong H_3(\overline{E_8 \times S^1 \setminus U}, \partial U;\Lambda)$. It is not too hard to see that F is isomorphic to the standard Lagrangian of H, generated by the f_i basis elements. Thus, all of the information about the formation as an element in $L_5(\mathbb{Z}[\mathbb{Z}])$ is contained in G. We shall describe G in terms of a basis e_i' which can be viewed as the upper half of a 16×16 matrix. For demonstrative purposes, we can describe F in the same way as the bottom half of the same matrix easily as $f_i' := f_i$ which corresponds to the matrix

$$\begin{bmatrix}\mathbf{0} & \mathrm{Id}\end{bmatrix}$$

where Id denotes the 8×8 identity matrix.

Lemma 6.1.3. *The matrix corresponding to G by describing a basis $\{e_i'\}$ for G in terms of the elements e_i and f_i is given below (the additional horizontal and vertical rules have been added for readability).*

$T-1$								$T+1$			1				
	$T-1$								$T+1$	1					
		$T-1$							T	$T+1$	1				
			$T-1$					T		T	$T+1$	1			
				$T-1$							T	$T+1$	1		
					$T-1$							T	$T+1$	1	
						$T-1$							T	$T+1$	1
							$T-1$							T	$T+1$

i.e. $e_1' = (T-1)e_1 + (T+1)f_1 + f_4$, $e_2' = (T-1)e_2 + (T+1)f_2 + f_3$ *etc.*

Remark 6.1.4. By construction (which will be seen below), the form should die on the e_i', but perhaps it is helpful to explicitly see this. Let λ denote the standard $\mathbb{Z}[\mathbb{Z}]$-valued intersection form on H. Then

$$\begin{aligned}
\lambda(e_1', e_1') &= \lambda((T-1)e_1, (T+1)f_1) + \lambda((T+1)f_1, (T-1)e_1)\\
&= (T - T^{-1}) + (T^{-1} - T) = 0,\\
\lambda(e_1', e_4') &= \lambda((T-1)e_1, Tf_1) + \lambda(f_4, (T-1)e_4) = (T-1)T^{-1} + T^{-1} - 1 = 0,
\end{aligned}$$

and all other cases are analogous.

Proof of Lemma 6.1.3. Since we are describing a basis for $G = H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda)$ we will work in the universal cover of $E_8 \times S^1$ which is $E_8 \times \mathbb{R}$. First arrange the embeddings $h_i \colon S^2 \times D^3 \hookrightarrow E_8 \times \mathbb{R}$ such that each $h_i(S^2 \times \{\mathrm{pt}\})$ lies in the slice $E_8 \times \{\frac{i-1}{8}\}$ and then write $e_i = h_i(S^2 \times \{\mathrm{pt}\})$ and similarly write $f_i = h_i(\{\mathrm{pt}\} \times S^2)$. Furthermore, we may assume that the projections p_i of e_i onto the E_8-factors give a basis for $H_2(E_8)$ corresponding to the basis used for (6.1.1). We will now describe eight distinct elements in G.

Let $A_i := p_i \times [\frac{i-1}{8}, 1 + \frac{i-1}{8}]$ be annuli in $E_8 \times \mathbb{R}$ and note these have $\partial A_i = (T-1)e_i$. The annuli A_i are not disjoint from U, but $A_i \cap U$ consists of disjoint 3-balls corresponding to the intersection form given in (6.1.1). However, note that because of how we chose to make the e_i disjoint in $E_8 \times \mathbb{R}$, if p_i and p_j have non-trivial intersection then A_i intersects U at e_j for $j > i$, at Te_j for $j < i$, and twice at e_j and Te_j for $j = i$. Remove all of these 3-balls from A_i to form the element A_i' which picks up extra boundary components as the boundaries of the removed 3-balls, which can be seen by taking the duals f_j for every e_j that appeared above. We now see that $\partial A_i' = e_i'$ as defined by the matrix in the statement of the lemma.

It remains to be seen that these eight elements generate the whole of G. Consider the following diagram, made out of the long exact sequence of the triple $(E_8 \times S^1, \overline{E_8 \times S^1 \setminus U}, \partial U)$ and the pair $(E_8 \times S^1, \partial U)$ (with Λ-coefficients suppressed).

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H_2(E_8 \times S^1) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H_2(\partial U) & & & & \\
 & & & & \partial \uparrow & & & & \\
0 & \longrightarrow & H_3(\overline{E_8 \times S^1 \setminus U}, \partial U) & \longrightarrow & H_3(E_8 \times S^1, \partial U) & \longrightarrow & H_3(U, \partial U) & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Further, the horizontal short exact sequence splits via the map on homology induced by the inclusion of pairs $(U, \partial U) \hookrightarrow (E_8 \times S^1, \partial U)$. Hence

$$H_3(E_8 \times S^1, \partial U) \cong H_3(\overline{E_8 \times S^1 \setminus U}, \partial U) \oplus H_3(U, \partial U).$$

Let $B \in H_3(\overline{E_8 \times S^1 \setminus U}, \partial U)$. Write $\partial B = \sum_{k=1,\dots,8} \lambda_k e_k + \mu_k f_k$. Further assume

$\lambda_k = (T-1)\overline{\lambda}_k$ for all k (for some $\overline{\lambda}_k \in \Lambda$), then

$$\begin{aligned}\partial(B - \sum_k \overline{\lambda}_k \overline{A}_k) &= \sum_k (T-1)\overline{\lambda}_k e_k + \mu_k f_k - \sum_k \overline{\lambda}_k \partial \overline{A}_k \\ &= \sum_k \overline{\mu}_k f_k\end{aligned}$$

for some $\overline{\mu}_k \in \Lambda$. Let C_k for $k = 1, \ldots, 8$ denote the basis for $H_3(U, \partial U)$ such that $\partial C_k = f_k$. Then

$$\partial(B - \sum_k \overline{\lambda}_k \overline{A}_k - \sum_k \overline{\mu}_k C_k) = \sum_k \overline{\mu}_k f_k - \overline{\mu}_k f_k = 0,$$

and hence the injectivity of ∂ implies that $B - \sum_k \overline{\lambda}_k \overline{A}_k - \sum_k \overline{\mu}_k C_k = 0 \in H_3(E_8 \times S^1, \partial U)$. Since this group splits as a direct sum, we see that B can be written as a Λ-linear combination of the $\overline{A}_i$.

We now claim that the assumption that $\lambda_k = (T-1)\overline{\lambda}_k$ holds for all B, which will complete the proof. Consider the following commutative diagram (where we are explicit about the coefficients).

$$\begin{array}{ccccccc} H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda) & \longrightarrow & H_3(E_8 \times S^1, \partial U; \Lambda) & \longrightarrow & H_2(\partial U; \Lambda) & \longrightarrow & \langle e_i \rangle_\Lambda \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \mathbb{Z}) & \longrightarrow & H_3(E_8 \times S^1, \partial U; \mathbb{Z}) & \longrightarrow & H_2(\partial U; \mathbb{Z}) & \longrightarrow & \langle e_i \rangle_\mathbb{Z} \end{array}$$

Here the vertical maps are the augmentation maps (given by setting $T = 1$). To prove the claim, it suffices to show that mapping any element $B \in H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda)$ horizontally along the diagram and vertically down to $\langle e_i \rangle_\mathbb{Z}$ gives the zero element. By considering the long exact sequence of the pair $(E_8 \times S^1, \partial U)$ with $\mathbb{Z}$-coefficients, one can see that the map $H_3(E_8 \times S^1, \partial U; \mathbb{Z}) \to H_2(\partial U; \mathbb{Z})$ only hits the subgroup generated by the f_i, and hence the composition of the final two lower horizontal maps in the diagram is the zero map. This completes the proof of the claim, and hence the proof of the whole lemma. □

§ 6.1.2 | Constructing the specific generator

The aim of this subsection is to take the generator for $L_5(\mathbb{Z}[\mathbb{Z}])$ that we constructed in the last subsection and show that it can be realised algebraically by a much smaller matrix. To do this, we will perform a sequence of row and column operations on the matrix from Lemma 6.1.3.

Let $\Sigma_i = \frac{1-T^{i+1}}{1-T} = 1 + T + \cdots + T^i$ and let $T' := T - 1$. We then perform the following sequence of row and column operations on the 8×16 matrix from Lemma 6.1.3, though we have given the initial matrix again below. Naturally, empty spaces denote zeroes but we have particularly noted zeroes when they have first appeared from the previous stage of row or column operations. We will sometimes use $\star$ to denote an entry whose value is too lengthy to succinctly state but whose precise value is not important to the calculation. We also give the row and column operations used to go between each step. For example, the notation

$$R_n \to R_n + \Sigma_k R_{n-1}$$

means add Σ_k times the $(n-1)$th row to the nth row. Similarly, we denote the nth column by C_n.

We now begin the operations.

$$\left[\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T+1 & & & 1 & & & & \\
 & T' & & & & & & & & T+1 & 1 & & & & & \\
 & & T' & & & & & & & T & T+1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
 & & & & & & & T' & & & & & & & T & T+1
\end{array}\right]$$

$$R_8 \to R_8 - \Sigma_1 R_7$$

$$\left[\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T+1 & & & 1 & & & & \\
 & T' & & & & & & & & T+1 & 1 & & & & & \\
 & & T' & & & & & & & T & T+1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
 & & & & & & -\Sigma_1 T' & T' & & & & & & -T\Sigma_1 & -\Sigma_2 & 0
\end{array}\right]$$

$$R_8 \to R_8 + \Sigma_2 R_6 - \Sigma_3 R_5 + \Sigma_4 R_4 - \Sigma_5 R_3 + \Sigma_6 R_2$$

$$\left[\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T+1 & & & 1 & & & & \\
 & T' & & & & & & & & T+1 & 1 & & & & & \\
 & & T' & & & & & & & T & T+1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
 & \Sigma_6 T' & -\Sigma_5 T' & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & T\Sigma_4 & \Sigma_7 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right]$$

$$R_3 \to R_3 - \Sigma_1 R_2$$

T'								$T+1$			1				
	T'								$T+1$	1					
	$-\Sigma_1 T'$	T'							Σ_2	0	1				
			T'					T		T	$T+1$	1			
				T'							T	$T+1$	1		
					T'							T	$T+1$	1	
						T'							T	$T+1$	1
	$\Sigma_6 T'$	$-\Sigma_5 T'$	$\Sigma_4 T'$	$-\Sigma_3 T'$	$\Sigma_2 T'$	$-\Sigma_1 T'$	T'	$T\Sigma_4$	Σ_7						

$$R_1 \to R_1 - R_3$$

T'	$\Sigma_1 T'$	$-T'$						$T+1$	Σ_2		0				
	T'								$T+1$	1					
	$-\Sigma_1 T'$	T'							$-\Sigma_2$		1				
			T'					T		T	$T+1$	1			
				T'							T	$T+1$	1		
					T'							T	$T+1$	1	
						T'							T	$T+1$	1
	$\Sigma_6 T'$	$-\Sigma_5 T'$	$\Sigma_4 T'$	$-\Sigma_3 T'$	$\Sigma_2 T'$	$-\Sigma_1 T'$	T'	$T\Sigma_4$	Σ_7						

$$C_1 \to C_1 + T^{-1} C_2$$

$$C_{10} \to C_{10} - T C_9$$

$T'(2+T^{-1})$	$\Sigma_1 T'$	$-T'$						$T+1$	1						
$T'T^{-1}$	T'								$T+1$	1					
$-T'T^{-1}\Sigma_1$	$-\Sigma_1 T'$	T'							$-\Sigma_2$		1				
			T'					T	$-T^2$	T	$T+1$	1			
				T'							T	$T+1$	1		
					T'							T	$T+1$	1	
						T'							T	$T+1$	1
$T'T^{-1}\Sigma_6$	$\Sigma_6 T'$	$-\Sigma_5 T'$	$\Sigma_4 T'$	$-\Sigma_3 T'$	$\Sigma_2 T'$	$-\Sigma_1 T'$	T'	$T\Sigma_4$	$\Sigma_7 - T^2\Sigma_4$						

$$R_8 \to R_8 - (\Sigma_7 - T^2\Sigma_4) R_1$$

$T'(2+T^{-1})$	$\Sigma_1 T'$	$-T'$						$T+1$	1						
$T'T^{-1}$	T'								$T+1$	1					
$-T^{-1}\Sigma_1$	$-\Sigma_1 T'$	T'							$-\Sigma_2$		1				
			T'					T	$-T^2$	T	$T+1$	1			
				T'							T	$T+1$	1		
					T'							T	$T+1$	1	
						T'							T	$T+1$	1
$\beta(T)$	$\star$	$\star$	$\Sigma_4 T'$	$-\Sigma_3 T'$	$\Sigma_2 T'$	$-\Sigma_1 T'$	T'	$\alpha(T)$	0						

0	0	0						0	1						
0	0								0	1					
0	0	0							0		1				
			0					0	0	0	0	1			
				0							0	0	1		
					0							0	0	1	
						0							0	0	1
$\beta(T)$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\alpha(T)$							

The last step uses the 1s on the right of the matrix to erase all non-zero entries to the left of them.

This concludes the matrix operations. The polynomials given in the last two matrices are defined as[2]

$$\begin{aligned}
\alpha(T) &= -1 - T + T^3 + T^4 + T^5 - T^7 - T^8, \\
\beta(T) &= (T-1)(-2 - T + T^2 + T^3 + T^4 + T^5 - T^6 - 2T^7).
\end{aligned}$$

We conclude that we can represent the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ by a 2-dimensional form H over Λ, and a pair of lagrangians F' and G' given by the matrix

$$M := \begin{array}{|c||c|} \hline \alpha(T) & \beta(T) \\ \hline 0 & 1 \\ \hline \end{array}.$$

Note that although we have used a matrix to encode the information about the lagrangians, this matrix does not correspond to the automorphism of the form which sends F' to G'. By Wall [Wal70, Corollary 5.3.1] we know that such an automorphism exists, and we can write it as the following matrix

$$M' := \begin{bmatrix} \gamma(T) & \alpha(T) \\ \delta(T) & \beta(T) \end{bmatrix}.$$

where we know the $\alpha(T)$ and $\beta(T)$ are as above, but $\gamma(T)$ and $\delta(T)$ are unknown. We can, however, say something about the augmentation of this automorphism, which will be useful to us in Chapter 7.

Lemma 6.1.5. *The matrix M' augments to the matrix*

$$M'(1) := \begin{bmatrix} \gamma(1) & \alpha(1) \\ \delta(1) & \beta(1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Proof. First note that $\alpha(1) = -1$ and $\beta(1) = 0$. Now, since the matrix $M'(1)$ must represent an automorphism of the $\mathbb{Z}$ valued hyperbolic form, the values of $\gamma(1)$ and

[2] In the original note, Lee's $\alpha(T)$ differs from the $\alpha(T)$ here by multiplication by -1. As far as I can tell, this was a sign error in the original computation in the very last step of the computation, since Lee's computation agrees with the computation here until the very last step.

$\delta(1)$ are already determined. A simple calculation shows that these values are $\gamma(1) = 0$ and $\delta(1) = -1$. $\square$

Presumably it is also possible to compute the exact Laurent polynomials $\gamma(T)$ and $\delta(T)$, but we have not attempted to do so.

§ 6.1.3 | Interpretation

Wall realisation [Wal70, Theorem 6.5] and Cappell-Shaneson [CS71, Theorem 3.1] tells us that we can represent the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ in the following way. Let W_1 be the standard cobordism between $S^1 \times S^3$ and $(S^1 \times S^3)\#(S^2 \times S^2)$ and let W_2 be the reversed cobordism. Let $(W; \partial_0 W = S^1 \times S^3, \partial_1 W = S^1 \times S^3)$ be the cobordism formed by gluing these via a homeomorphism

$$\sigma\colon S^1 \times S^3 \# S^2 \times S^2 \to S^1 \times S^3 \# S^2 \times S^2$$

whose induced map on $H_2(S^1 \times S^3 \# S^2 \times S^2; \Lambda)$ is exactly the 2×2-matrix M' above. Such a homeomorphism exists by Stong-Wang [SW00, Theorem 2]. Then the degree-one normal map

$$f\colon (W; \partial_0 W, \partial_1 W) \to (S^1 \times S^3 \times I; S^1 \times S^3 \times \{0\}, S^1 \times S^3 \times \{1\})$$

has surgery obstruction $\theta(f)$ the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ defined by M'.

§ 6.2 | Proof of the stable realisation theorem

In this section we will prove Theorem 6.0.1 which shows that the Casson-Sullivan invariant can always be realised stably. The idea is to use Theorem 5.5.3 along with the following proposition, which relies on the work of developed in the previous section.

Proposition 6.2.1. *There exists a homeomorphism*

$$\sigma\colon (S^1 \times S^3)\#(S^2 \times S^2) \to (S^1 \times S^3)\#(S^2 \times S^2)$$

such that $\mathrm{cs}(\sigma)$ *is the generator of* $H^3(S^1 \times S^3; \mathbb{Z}/2)$*, the dual to the generator of* $H_1(S^1 \times S^3; \mathbb{Z}/2)$ *which is represented by* $\theta = S^1 \times \{\mathrm{pt}\}$*. Furthermore,* $\sigma|_{\nu\theta} = \mathrm{Id}_{\nu\theta}$*.*

We postpone the proof of this proposition until Section 6.2.2. Now we use Theorem 5.5.3 and Proposition 6.2.1 to prove the following theorem, from which Theorem 6.0.1 will immediately follow.

Theorem 6.2.2. *Let* $f_0\colon X_0 \to X_0'$ *be a homeomorphism of compact, connected, smooth, orientable 4-manifolds and define* $\eta_0 := \mathrm{cs}(f_0)$*. Let* $\eta \in H^3(X_0, \partial X_0; \mathbb{Z}/2)$*, and let* $\gamma \subset X$ *be a (framed) embedded curve dual to* $\eta - \eta_0 \in H^3(X_0, \partial X_0; \mathbb{Z}/2)$*. Then we have the following cases.*

(i) If $f_0|_{\nu\gamma}$ *is the identity map, then the connected-sum homeomorphism*

$$f_0 \#_{\gamma=\theta} \sigma\colon X_0 \#(S^2 \times S^2) \to X_0' \#(S^2 \times S^2)$$

is defined and $\mathrm{cs}(f\#_{\gamma=\theta}\sigma)=\eta$.

(ii) If $f_0|_{\nu\gamma}$ *is the Gluck twist map, then the connected-sum homeomorphism*

$$f_0\#_{\gamma=\theta}(\sigma\circ t)\colon X_0\#(S^2\times S^2)\to X_0'\#(S^2\times S^2),$$

where t *denotes the Gluck twist map extended onto the* $(S^2\times S^2)$*-summand, is defined and* $\mathrm{cs}(f_0\#_{\gamma=\theta}\sigma)=\eta$.

Proof (assuming Proposition 6.2.1). That we only have to consider the two cases above comes from the exposition at the beginning of Section 5.5. In both cases the connected-sum homeomorphism (as above) is defined by the last part of Proposition 6.2.1, and because there are natural diffeomorphisms

$$X_0\#_{\gamma=\theta}((S^1\times S^3)\#(S^2\times S^2))\cong X_0\#(S^2\times S^2)$$

and

$$X_0'\#_{(f_0)_*(\gamma)=\theta}((S^1\times S^3)\#(S^2\times S^2))\cong X_0'\#(S^2\times S^2).$$

Let Q be the map in the statement of Theorem 5.5.3 for the above decomposition. In case (i), by Theorem 5.5.3 and Proposition 6.2.1 we have

$$\mathrm{cs}(f_0\#_{\gamma=\theta}\sigma)=Q(\mathrm{cs}(f_0),(\mathrm{cs}(\sigma)))=\eta_0+(\eta-\eta_0)=\eta.$$

Similarly, in case (ii), by Proposition 5.4.2, Theorem 5.5.3 and Proposition 6.2.1 we have

$$\begin{aligned}\mathrm{cs}(f_0\#_{\gamma=\theta}(\sigma\circ t))&=Q(\mathrm{cs}(f_0),\mathrm{cs}(\sigma\circ t))\\&=Q(\mathrm{cs}(f_0),\mathrm{cs}(\sigma)+\mathrm{cs}(t))\\&=Q(\mathrm{cs}(f_0),\mathrm{cs}(\sigma))\\&=\eta_0+(\eta-\eta_0)=\eta.\end{aligned}$$

In the above formulae we have used the isomorphism

$$H^3(X_0,\partial X_0;\mathbb{Z}/2)\cong H^3(X_0\#(S^2\times S^2),\partial X_0;\mathbb{Z}/2)$$

induced by the map collapsing the $S^2\times S^2$ summand. □

Proof of Theorem 6.0.1. Using the isomorphism $H^3(X_0,\partial X_0;\mathbb{Z}/2)\cong H^3(X,\partial X;\mathbb{Z}/2)$ induced by collapsing the $S^2\times S^2$ summand, we can consider the given class $\eta\in H^3(X,\partial X;\mathbb{Z}/2)$ as an element $\eta\in H^3(X_0,\partial X_0;\mathbb{Z}/2)$. By assumption there exists a homeomorphism $f_0\colon X_0\to X_0'$, and applying Theorem 6.2.2 to the class η immediately gives the result. □

§ 6.2.1 | Surgery

Now we fundamentally make use of the result from Section 6.1, where we presented Ronnie Lee's proof that the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ could be represented by a 2×2 matrix. Using this, we receive the following corollary.

Corollary 6.2.3. *There exists a homeomorphism*

$$\sigma\colon (S^1 \times S^3)\#(S^2 \times S^2) \to (S^1 \times S^3)\#(S^2 \times S^2)$$

realising the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$. *More precisely, let* N *be the standard cobordism between* $S^1 \times S^3$ *and* $(S^1 \times S^3)\#(S^2 \times S^2)$. *Then the surgery problem*

$$W = (N \cup_\sigma -N) \to (S^1 \times S^3) \times I$$

has surgery obstruction the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$. *Furthermore,* $\sigma|_{\nu\theta} = \mathrm{Id}_{\nu\theta}$, *where* $\theta = S^1 \times \{\mathrm{pt}\} \subset S^1 \times S^3$.

Proof. By Section 6.1 we have a matrix M with entries in $\mathbb{Z}[\mathbb{Z}]$ which represents the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$. By [SW00, Theorem 2] there exists two pseudo-isotopy classes of self-homeomorphisms of $(S^1 \times S^3)\#(S^2 \times S^2)$ which induce the map M on $H_2((S^1 \times S^3)\#(S^2 \times S^2); \mathbb{Z}[\mathbb{Z}])$, one which preserves the spin structures on $S^1 \times S^3$ and one which swaps them. Let σ be a representative self-homeomorphism that fixes the spin structures. Since σ fixes the spin structures, σ can be isotoped such that $\sigma|_{\nu\theta} = \mathrm{Id}_{\nu\theta}$. It follows from [CS71, Theorem 3.1] that W has surgery obstruction the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$. □

§ 6.2.2 | Proof of Proposition 6.2.1

Throughout this subsection, let $\sigma\colon (S^1 \times S^3)\#(S^2 \times S^2) \to (S^1 \times S^3)\#(S^2 \times S^2)$ and W be as in Corollary 6.2.3. The aim is to prove Proposition 6.2.1 by showing that that this σ has $\mathrm{cs}(\sigma)$ the generator of $H^3((S^1 \times S^3)\#(S^2 \times S^2); \mathbb{Z}/2)$. This will be done by performing operations on W given in Corollary 6.2.3 and keeping track of what happens to the relative Kirby-Siebenmann invariant of the cobordism along the way. We begin with some lemmas.

In what follows, we will also need the following lemma.

Lemma 6.2.4. *Every homeomorphism* $f\colon S^1 \times S^3 \to S^1 \times S^3$ *is pseudo-smoothable and hence has* $\mathrm{cs}(f) = 0$.

Proof. Stong-Wang classified homeomorphisms of 4-manifolds M with $\pi_1(M) \cong \mathbb{Z}$ up to pseudo-isotopy [SW00, Theorem 2]. This directly gives us that there are four homeomorphisms on $S^1 \times S^3$ up to pseudo-isotopy represented by (1) the identity map; (2) conjugation on the S^1-factor composed with the reflection map on the S^3-factor; (3) the corresponding Gluck twist map $S^1 \times S^3 \to S^1 \times S^3$ (Equation (5.5.1)); and (4) the composition of the two previously stated non-trivial maps. All of these maps are clearly smooth, hence every self-homeomorphism of $S^1 \times S^3$ is pseudo-smoothable. □

Lemma 6.2.5. *Under the standard identification*

$$H^4(W,\partial W;\mathbb{Z}/2)\cong H^4(M_\sigma,\partial M_\sigma;\mathbb{Z}/2)\xrightarrow{\varpi^{-1}} H^3((S^1\times S^3)\#(S^2\times S^2);\mathbb{Z}/2)$$

(*see Definition 5.2.9*) *we have that* $\operatorname{ks}(W,\partial W)=\varpi\operatorname{cs}(\sigma)$ *where* M_σ *denotes the mapping cylinder.*

Proof. Note that N is smoothable relative to the standard smooth structure on the boundary. It is clear that we have a homeomorphism relative to the boundary

$$W\approx N\cup_{(S^1\times S^3)\#(S^2\times S^2)}\Big(((S^1\times S^3)\#(S^2\times S^2))\times I\Big)\cup_\sigma -N.$$

i.e. inserting a product $(S^1\times S^3\#S^2\times S^2)\times I$ to the right end of N does not change the relative homeomorphism type of W. We now see that the identification

$$H^4(W,\partial W;\mathbb{Z}/2)\xrightarrow{\cong} H^4(M_\sigma,\partial M_\sigma;\mathbb{Z}/2)$$

gives $\operatorname{ks}(W,\partial W)=\operatorname{ks}(M_\sigma)=\varpi\operatorname{cs}(\sigma)$ where M_σ is the mapping cylinder of σ (note that here we write "=" as there is only one isomorphism between groups isomorphic to $\mathbb{Z}/2$.). □

Lemma 6.2.6 ([FQ90, Proof of 11.6A])**.** *Let* $S^1\times E_8\to S^1\times S^4$ *be the surgery problem which has surgery obstruction the inverse of the generator of* $L_5(\mathbb{Z}[\mathbb{Z}])$*, let* $\gamma\subset W$ *be an embedded curve representing the generator of* $H_1(W)$*, and let* γ' *be the embedded curve* $S^1\times\{\mathrm{pt}\}\subset S^1\times E_8$*. Then the connected-sum surgery problem*

$$W\#_{\gamma=\gamma'}(S^1\times E_8)\to(S^1\times S^3)\times I$$

has vanishing surgery obstruction.

Remark 6.2.7. Note that the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ being representable by a surgery problem $S^1\times E_8\to S^1\times S^4$ (as in Lemma 6.2.6) follows from Shaneson splitting [Sha69, Theorem 5.1].

We can now prove the proposition.

Proof of Proposition 6.2.1. Start with the cobordism W and modify the surgery obstruction to be trivial using Lemma 6.2.6. Since the result has vanishing surgery obstruction, we can surger it relative to the boundary to an s-cobordism W'. Since $\pi_1(W')\cong\pi_1(S^1\times S^3)\cong\mathbb{Z}$ is good we can use the s-cobordism theorem [FQ90, Theorem 7.1A], which gives that W' is homeomorphic to the mapping cylinder M_f for some homeomorphism $f\colon S^1\times S^3\to S^1\times S^3$. By Lemma 6.2.4, $\operatorname{cs}(f)=0$ and hence $\operatorname{ks}(W',\partial W')=0$.

We now need to keep track of how we modified the Kirby-Siebenmann invariant throughout this process. The main tool for doing so will be the long exact sequence of the triple (see Equation (5.5.3)) with $\mathbb{Z}/2$-coefficients suppressed throughout. When we used Lemma 6.2.6 to kill the surgery obstruction, the connected-sum over a circle was induced by a relative cobordism C between $W \sqcup (E_8 \times S^1)$ and $W \#_{\gamma=\gamma'} (S^1 \times E_8)$. Let $W_\# := W \#_{\gamma=\gamma'} (S^1 \times E_8)$. This relative cobordism C is formed by attaching a single 1-handle (at one point on γ and at one point on γ') and then a single 2-handle which attaches by going along γ, then the 1-handle, then γ', and then back along the 1-handle. From the long exact sequence of the triple $(C, W \sqcup S^1 \times E_8, \partial W)$ we have the sequence

$$H^4(C, W \sqcup S^1 \times E_8) \to H^4(C, \partial W) \to H^4(W \sqcup S^1 \times E_8, \partial W) \to H^5(C, W \sqcup S^1 \times E_8).$$

The outer groups $H^4(C, W \sqcup S^1 \times E_8)$ and $H^5(C, W \sqcup S^1 \times E_8)$ must both vanish since the cobordism C was made by only attaching 1- and 2-handles. Hence we get an isomorphism

$$H^4(C, \partial W) \xrightarrow{\cong} H^4(W \sqcup S^1 \times E_8, \partial W) \cong H^4(W, \partial W) \oplus H^4(S^1 \times E_8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{6.2.1}$$

By considering the long exact sequence of the triple $(C, W_\#, \partial W_\#)$ we have the following commutative diagram

$$\begin{array}{ccccc} H^4(C, W_\#) & \longrightarrow & H^4(C, \partial W_\#) & \longrightarrow & H^4(W_\#, \partial W_\#) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}/2 & \xrightarrow{1 \mapsto (1,1)} & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xrightarrow{(a,b) \mapsto a+b} & \mathbb{Z}/2. \end{array}$$

The leftmost vertical isomorphism again comes from the relative handle decomposition of C, and the rightmost vertical isomorphism is clear by the definition of $W_\#$. The middle vertical isomorphism comes from Equation (6.2.1) and the fact that $H^*(C, \partial W_\#) \cong H^*(C, \partial W)$. The leftmost horizontal map can be seen to be the diagonal map since the generator of $H^4(C, W_\#)$ is Poincaré-Lefschetz dual to the annulus with boundaries homologous to the generators of $H_1(W)$ and $H_1(S^1 \times E_8)$. The rightmost horizontal map is then given by exactness. Using this and naturality of the Kirby-Siebenmann invariant, we can deduce that

$$\mathrm{ks}(W_\#, \partial W_\#) = \mathrm{ks}(W, \partial W) + \mathrm{ks}(S^1 \times E_8),$$

where we can naturally identify the groups as there is only one isomorphism between groups isomorphic to $\mathbb{Z}/2$.

Using a similar argument, one can see that the surgeries used to surger $W_\#$ to the s-cobordism W' do not alter the Kirby-Siebenmann invariant (the relative handle decomposition for the relative cobordism given by the trace of any given surgery consists of only a single handle, so the computation is greatly simplified). Hence, since by naturality $\mathrm{ks}(S^1 \times E_8) = 1$,

$$\mathrm{ks}(W', \partial W') = \mathrm{ks}(W_\#, \partial W_\#) = \mathrm{ks}(W, \partial W) + \mathrm{ks}(S^1 \times E_8) = \mathrm{ks}(W, \partial W) + 1$$

As we already concluded that $\mathrm{ks}(W', \partial W') = 0 \in \mathbb{Z}/2$, this implies that $\mathrm{ks}(W, \partial W) = 1$, and hence, by Lemma 6.2.5, $\mathrm{cs}(\sigma)$ is the generator of $H^3((S^1 \times S^3)\#(S^2 \times S^2); \mathbb{Z}/2)$, as claimed. □

§ 6.2.3 | An interesting example

We can use the objects and tools that we have developed so far to illustrate an interesting example involving the Casson-Sullivan invariant that demonstrates its dependence on smooth structures, as is expected by Section 5.3. This will also prove Corollary 6.0.2.

Example 6.2.8. Let $X = (S^1 \times S^3)\#(S^1 \times S^3)\#(S^2 \times S^2)$. Let $f := \mathrm{Id}_{S^1 \times S^3} \# \sigma \colon X \to X$ be a homeomorphism where σ denotes the homeomorphism constructed by Corollary 6.2.3, and let $g \colon X \to X$ be the homeomorphism which swaps the $S^1 \times S^3$ summands and leaves the $S^2 \times S^2$ summand fixed.

Let $\mathscr{S}$ denote the standard smooth structure on X and (in an abuse of notation) also denote the induced formal smooth structure (see Remark 5.2.6). Then $g \colon X_{\mathscr{S}} \to X_{\mathscr{S}}$ is (isotopic to) a diffeomorphism, and hence $\mathrm{cs}(g) = 0$ with respect to the smooth structure $\mathscr{S}$. However, let $f^*(\mathscr{S})$ be the smooth structure on X induced by f. Using Proposition 5.3.2 we can compute the Casson-Sullivan invariant of g with respect to the smooth structure $f^*(\mathscr{S})$. As in Proposition 5.3.2, let $a \in [X, \mathrm{TOP}/\mathrm{O}]$ be the unique element such that $a \cdot \mathscr{S} = f^*(\mathscr{S})$. We then have that

$$\begin{aligned}\mathrm{cs}(g) &= \delta(g, f^*(\mathscr{S}))\\ &= a + g^*(a) + \delta(g, \mathscr{S})\\ &= a + g^*(a)\\ &= \delta(f, \mathscr{S}) + g^*(\delta(f, \mathscr{S}))\\ &= \mathrm{cs}(f) + g^*(\mathrm{cs}(f)),\end{aligned}$$

which is equal to the element $(1, 1) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong H^3(X; \mathbb{Z}/2)$ by Theorem 5.6.1 and Proposition 6.2.1. So g is no longer smoothable with respect to the smooth structure $f^*(\mathscr{S})$.

Corollary 6.0.2 follows immediately from this example.

Proof of Corollary 6.0.2. Take $\mathscr{S}' := f^*(\mathscr{S})$. Let $X_{\mathscr{S}}$ denote X with the standard smooth structure, and let $X_{\mathscr{S}'}$ denote X with the smooth structure $\mathscr{S}'$. By definition, $f \colon X_{\mathscr{S}} \to X_{\mathscr{S}'}$ is a diffeomorphism, so the two smooth structures are diffeomorphic (see Definition 2.1.4). By the calculation in Example 6.2.8, g is not stably pseudo-smoothable with respect to the smooth structure $\mathscr{S}'$, since its Casson-Sullivan invariant is non-trivial. □

Chapter 7

Stable isotopy of surfaces

This chapter is devoted to proving Theorem 1.2.10. We begin with a definition.

Definition 7.0.1. Let X be a connected, compact, orientable smooth 4-manifold and let $\Sigma_1, \Sigma_2 \subset X$ be a pair of smoothly embedded surfaces, such that $\partial\Sigma_1 = \partial\Sigma_2 = L \subset \partial X$ a fixed link in ∂X (which may be disconnected). We say that Σ_1 and Σ_2 are *topologically isotopic* {smoothly isotopic} if there exists a homeomorphism {diffeomorphism} of pairs $F\colon (X, \Sigma_1) \to (X, \Sigma_2)$ such that F is isotopic {smoothly isotopic} to the identity. We say that Σ_1 and Σ_2 are *externally stably smoothly isotopic* if there exists $n \geq 0$ such that Σ_1 and Σ_2 become smoothly isotopic in $X\#(\#^n S^2 \times S^2)$, where we perform the connected-sums in the complement of $\Sigma_1 \cup \Sigma_2$.

Now we restate Theorem 7.0.2 from the introduction.

Theorem 7.0.2. *Let X be a connected, compact, simply-connected, smooth 4-manifold and let $\Sigma_1, \Sigma_2 \subset X$ be a pair of smoothly, properly embedded, orientable surfaces which are topologically isotopic relative to their boundaries. Then Σ_1 and Σ_2 are externally stably smoothly isotopic.*

The idea of proving Theorem 7.0.2 is as follows: first, we find a stable diffeomorphism between the exteriors of the surfaces. Then, we show that we can modify this diffeomorphism to find one such that when we glue back in the tubular neighbourhoods of the surfaces, we have a diffeomorphism which is the identity on $H_2(X;\mathbb{Z})$. Then we use the following result of Saeki and Orson-Powell which says when a diffeomorphism of a simply-connected 4-manifold is stably isotopic to the identity.

Theorem 7.0.3 ([Sae06, OP23])**.** *Let X be a connected, compact, simply-connected smooth 4-manifold with (potentially disconnected) boundary ∂X and let $f\colon X \to X$ be a diffeomorphism (restricting to the identity on ∂X) such that the variation induced by f is trivial, and such that the induced map on relative spin structures (see [OP23, Definition 2.5]) is trivial. Then f is stably smoothly isotopic to the identity.*

The above theorem crucially relies on the theorem of Quinn [Qui86] that smooth pseudo-isotopy implies stable smooth isotopy (c.f. [Gab22, Section 2], and [GGH+23]).

Since in Theorem 7.0.2 we assume that our surfaces are topologically isotopic, this means that we have a homeomorphism of the surface exteriors, and we will take this as our starting point. First, however, we need to arrange that our homeomorphism is smooth near the surfaces.

Lemma 7.0.4. *Let X, Σ_1 and Σ_2 be as in the statement of Theorem 7.0.2 and let $\widehat{G}\colon X \to X$ be a homeomorphism which sends Σ_1 to Σ_2. Then $\widehat{G}$ is isotopic relative $\partial X \cup \Sigma_1$ to a homeomorphism which sends $\nu\Sigma_1$ to $\nu\Sigma_2$ by a diffeomorphism.*

Proof. Let $U := \widehat{G}(\nu\Sigma_1)$. We begin by isotoping $\widehat{G}|_{\Sigma_1}$ to a diffeomorphism, which is always possible since homeomorphisms of surfaces are isotopic to diffeomorphisms [Eps66] (see also [Hat14]). This isotopy extends to $\widehat{G}$ by extending it first to a tubular neighbourhood (perform the isotopy less and less as you extend radially from Σ_1) and then as the constant isotopy on the complement of the tubular neighbourhood of Σ_1 (of course, it was clear, by the isotopy extension theorem [EK71], that this extended, but we can see the extension explicitly and easily in this case). Denote the result of this isotopy still by $\widehat{G}$. By the uniqueness of topological tubular neighbourhoods [FQ90, Chapter 9.3], U and $\nu\Sigma_2$ are isotopic by an isotopy that fixes Σ_2, and hence we can isotope $\widehat{G}$ relative to $\partial X \cup \Sigma_1$ such that $\widehat{G}(\nu\Sigma_1) = \nu\Sigma_2$ as bundles. We now smooth the map on the fibres of the normal bundles relative to the 0-section Σ_1, which we can do since any map $\Sigma_1 \to \mathrm{O}(2)$ can be isotoped to a smooth map. The result of this isotopy now sends $\nu\Sigma_1$ to $\nu\Sigma_2$ via a diffeomorphism. □

Remark 7.0.5. It should be noted that one cannot always smooth homeomorphisms of 4-manifolds near arbitrary codimension 2 submanifolds. In general, one can only smooth a homeomorphism near a codimension 2 submanifold after a small topological isotopy (see [FQ90, Theorem 8.1A]). Lemma 7.0.4 does not contradict this since we have a much stronger hypothesis (it is a self-homemorphism of a smooth manifold and our homeomorphism already maps the submanifold in question to another smooth submanifold).

The following lemma will handle most of the technical aspects of the proof of Theorem 7.0.2.

Lemma 7.0.6. *Let X, Σ_1 and Σ_2 be as in the statement of Theorem 7.0.2 and let $G\colon X \setminus \nu\Sigma_1 \to X \setminus \nu\Sigma_2$ be the homeomorphism of the surface exteriors and $\widehat{G}\colon X \to X$ the extension of G. Then for some $k \geq 0$ there exists a diffeomorphism*

$$F\colon (X \setminus \nu\Sigma_1)\#(\#^k S^2 \times S^2) \to (X \setminus \nu\Sigma_2)\#(\#^k S^2 \times S^2)$$

which restricts to the identity on ∂X. Furthermore, F extends to a diffeomorphism

$$\widehat{F}\colon X\#(\#^k S^2 \times S^2) \to X\#(\#^k S^2 \times S^2),$$

and the induced maps on second homology fit into the following commutative diagram.

$$\begin{array}{ccc} H_2(X\setminus\nu\Sigma_1)\oplus\mathbb{Z}^2\oplus\mathbb{Z}^{2k-2} & \xrightarrow{(G_*,A,\mathrm{Id})} & H_2(X\setminus\nu\Sigma_2)\oplus\mathbb{Z}^2\oplus\mathbb{Z}^{2k-2} \\ \downarrow\cong & & \downarrow\cong \\ H_2((X\setminus\nu\Sigma_1)\#(\#^kS^2\times S^2)) & \xrightarrow{F_*} & H_2((X\setminus\nu\Sigma_2)\#(\#^kS^2\times S^2)) \\ \downarrow(i_1)_* & & \downarrow(i_2)_* \\ H_2(X\#(\#^kS^2\times S^2)) & \xrightarrow{(\widehat{F})_*} & H_2(X\#(\#^kS^2\times S^2)) \\ \downarrow\cong & & \downarrow\cong \\ H_2(X)\oplus\mathbb{Z}^2\oplus\mathbb{Z}^{2k-2} & \xrightarrow{(\mathrm{Id},A,\mathrm{Id})} & H_2(X)\oplus\mathbb{Z}^2\oplus\mathbb{Z}^{2k-2} \end{array} \tag{7.0.1}$$

where i_1 and i_2 denote the inclusions of the exteriors, and $A\colon \mathbb{Z}^2\to\mathbb{Z}^2$ is either the identity map if $\mathrm{cs}(G)=0$, or is given by the map sending $(x,y)\mapsto(-y,-x)$ if $\mathrm{cs}(G)\neq 0$. Furthermore, $\widehat{F}$ is topologically pseudo-isotopic to a map which differs from the stabilisation of $\widehat{G}$ only on a neighbourhood of a curve in $X\setminus\nu\Sigma_1$ and a single $S^2\times S^2$ summand.

Proof. We assume, by Lemma 7.0.4, that $\widehat{G}\colon X\to X$ already restricts to a diffeomorphism of the tubular neighbourhoods $\widehat{G}|_{\nu\Sigma_1}\colon\nu\Sigma_1\to\nu\Sigma_2$.

We begin by showing the existence of a stable diffeomorphism of the exteriors, and that the top square in Equation (7.0.1) commutes. Assume that $\mathrm{cs}(G)\neq 0$, and let $\gamma\subset X\setminus\nu\Sigma_1$ be a framed embedded curve such that $[\gamma]\in H_1(X\setminus\nu\Sigma_1;\mathbb{Z}/2)$ is dual to $\mathrm{cs}(G)$. By Theorem 6.2.2, the connected-sum homeomorphism (using the notation therein)

$$G' := G\#_{\gamma=\theta}\widehat{\sigma}\colon (X\setminus\nu\Sigma_1)\#(S^2\times S^2)\to(X\setminus\nu\Sigma_2)\#(S^2\times S^2)$$

has $\mathrm{cs}(G')=\mathrm{cs}(G)+\mathrm{cs}(G)=0$, where $\widehat{\sigma}=\sigma$ or $\sigma\circ t$, depending on G.

By Proposition 5.4.5, G' is stably pseudo-isotopic to a diffeomorphism

$$F\colon (X\setminus\nu\Sigma_1)\#(\#^kS^2\times S^2)\to(X\setminus\nu\Sigma_2)\#(\#^kS^2\times S^2)$$

for some $k\geq 1$. By Lemma 6.1.5, $\widehat{\sigma}$ induces the map sending $(x,y)\mapsto(-y,-x)$ on $H_2((S^1\times S^3)\#(S^2\times S^2))$ and hence the top square in Equation (7.0.1) commutes.

If $\mathrm{cs}(G)=0$ then we may immediately apply Proposition 5.4.5 and similarly obtain a diffeomorphism F, though potentially needing no stabilisations, and the top half of Equation (7.0.1) commutes with A given by the identity map.

Now F extends to a diffeomorphism on $X\#(\#^kS^2\times S^2)$ since we already assumed that $\widehat{G}$ restricted to a diffeomorphism $\nu\Sigma_1\to\nu\Sigma_2$. More specifically, we can fill back in the tubular neighbourhoods of the surfaces Σ_1 and Σ_2 to obtain a diffeomorphism

$\widehat{F}$ which fits into the following diagram.

$$\begin{array}{ccc} (X \setminus \nu\Sigma_1)\#(\#^k S^2 \times S^2) & \xrightarrow{F} & (X \setminus \nu\Sigma_2)\#(\#^k S^2 \times S^2) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ X\#(\#^k S^2 \times S^2) & \xrightarrow{\widehat{F}} & X\#(\#^k S^2 \times S^2) \end{array}$$

This means that the middle square in Equation (7.0.1) must commute, and the commutativity of the bottom square then follows immediately. The final statement is clear by the construction. □

Proof of Theorem 7.0.2. By Lemma 7.0.6, we have that for some $k \geq 0$ there exists a diffeomorphism

$$\widehat{F}\colon X\#(\#^k S^2 \times S^2) \to X\#(\#^k S^2 \times S^2)$$

such that on second homology it induces the map

$$(\mathrm{Id}, A, \mathrm{Id})\colon H_2(X) \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^{2k-2} \to H_2(X) \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^{2k-2}$$

where A is either the identity map or the map sending $(x, y) \mapsto (-y, -x)$. Since $\widehat{F}$ is an extension of a diffeomorphism of the exteriors, it must send Σ_1 to Σ_2.

We want to apply Theorem 7.0.3, and so we need to show that the variation induced by $\widehat{F}$, as well as the induced map on relative spin structures, is trivial. By the last statement of Lemma 7.0.6, the variation induced by $\widehat{F}$ can only differ by the variation induced by $\widehat{G}$ by its action on relative homology classes which cannot be represented by a relative surface disjoint from the neighbourhood of a curve union a single $(S^2 \times S^2)$-connected-summand. Such a class $x \in H_2(X\#(\#^k S^2 \times S^2), \partial)$ can be represented as $x = x' + x''$, where x' is represented by a relative surface disjoint from a curve union a single $(S^2 \times S^2)$-connected-summand, and x'' is a (potentially trivial) homology class on that $(S^2 \times S^2)$-connected-summand. Since the variation induced by $\widehat{G}$ is topologically isotopic to the identity, its induced variation is trivial, and hence the action of the variation induced by $\widehat{F}$ on all relative classes disjoint from a curve union a single $(S^2 \times S^2)$-connected-summand is trivial. Putting these facts together, this means that the variation induced by $\widehat{F}$ can only differ from the trivial variation if its induced map on homology is non-trivial.

If A is the identity map, then $\widehat{F}$ acts trivially on homology, and hence (by the above) its induced variation is trivial. If A is not the identity map, then post-compose $\widehat{F}$ with the map

$$a := \mathrm{Id}\,\#a'\#\,\mathrm{Id}\colon X\#(S^2 \times S^2)\#(\#^k S^2 \times S^2) \to X\#(S^2 \times S^2)\#(\#^k S^2 \times S^2),$$

where $a'\colon S^2 \times S^2 \to S^2 \times S^2$ is the map defined as the antipodal map on both S^2-factors, composed with the diffeomorphism that swaps the two S^2-factors, and obtain an orientation-preserving diffeomorphism $a \circ \widehat{F}$. This new diffeomorphism still sends

Σ_1 to Σ_2, since a is supported away from Σ_1, but now induces the trivial map on homology. In either case, this means the map we have created now induces the trivial variation.

That the induced map on relative spin structures by $\widehat{F}$ is trivial follows again from the last statement in Lemma 7.0.6 and that every arc between two distinct boundary components of X can be made disjoint from a curve union a single $(S^2 \times S^2)$-summand.

Now apply Theorem 7.0.3 to obtain that $\widehat{F}$ is stably smoothly isotopic to the identity, and hence Σ_1 and Σ_2 are stably smoothly isotopic. □

As was mentioned in Remark 1.2.11, Theorem 7.0.2 has the following consequence when paired with certain results concerning when embedded surfaces are (stably) topologically isotopic.

Corollary 7.0.7. *Let X be as in Theorem 7.0.2. Let Σ_1, Σ_2 be a pair of homologous, smoothly embedded surfaces in X with the same genus and same boundaries, such that $\pi_1(X \setminus \nu\Sigma_1) \cong \pi_1(X \setminus \nu\Sigma_2)$. Then if any of the following conditions are satisfied, the surfaces are stably smoothly isotopic relative to their boundaries (below $b_2(X)$ denotes the second Betti number of X and $\operatorname{sig}(X)$ denotes the signature of X).*

(i) Σ_1 *and* Σ_2 *are both spheres and* $\pi_1(X \setminus \nu\Sigma_1) \cong \mathbb{Z}/d$ *for some* $d \geq 0$.

(ii) $\pi_1(X \setminus \nu\Sigma_1)$ *is trivial.*

(iii) $\pi_1(X \setminus \nu\Sigma_1) \cong \mathbb{Z}$ *and* $b_2(X) \geq |\operatorname{sig}(X)| + 6$.

(iv) $\pi_1(X \setminus \nu\Sigma_1) \cong \mathbb{Z}/d$ *for some* $d \geq 2$ *and* $b_2(X) > |\operatorname{sig}(X)| + 2$ *and the genus of* Σ_1 *is not the minimal genus needed to represent that homology class among surfaces with the same fundamental group of the exterior.*

Proof. If case (i) is satisfied then [HK93b, Theorem 4.8] (or [LW90, Corollary 1.3] if d is odd and X is closed) shows that the surfaces are stably topologically isotopic. Then Theorem 7.0.2 applied to the stabilisation shows that the surfaces are stably smoothly isotopic. If any of the other cases are satisfied, then [Sun15b, Theorem 7.1, 7.2, 7.4] shows that the surfaces are topologically isotopic, and then applying Theorem 7.0.2 shows that they are stably smoothly isotopic. □

Chapter 8

Unstable realisation of the Casson-Sullivan invariant

In this chapter we aim to realise the Casson-Sullivan invariant unstably in some cases. The main result that we will prove is the following, which was Theorem 1.2.6 from the introduction.

Theorem 8.0.1. *Let X be a compact, connected, smooth, orientable 4-manifold with $\pi_1(X)$ a good group such that X satisfies the Casson-Sullivan realisability condition (Definition 8.3.1). Then for every class $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$ there exists a homeomorphism $f\colon X \to X$ with $\mathrm{cs}(f) = \eta$.*

The Casson-Sullivan realisability condition will be given in Definition 8.3.1. We will also prove the following theorem, which was Theorem 1.2.8 from the introduction.

Theorem 8.0.2. *Let X be a compact, connected, smooth, orientable 4-manifold with $\pi_1(X)$ a good group such that X satisfies the Casson-Sullivan realisability condition (Definition 8.3.1). Then there exists a family of homeomorphisms $\{f_\eta\colon X \to X \mid 0 \neq \eta \in H^3(X, \partial X; \mathbb{Z}/2)\}$ all distinct up to pseudo-isotopy (relative to the boundary) such that each element f_η is not stably pseudo-smoothable but each f_η is homotopic to the identity map.*

These are the most subtle non-smoothable homeomorphisms that we can produce, in the sense that they are "homotopically smoothable" but not actually smoothable. Compare this to the non-smoothable homeomorphisms produced in Theorem 4.0.1, which were trivial on homology but were still homotopically non-trivial, being detected by Poincaré variations.

§ 8.0.1 | Chapter outline

In Section 8.1 we will introduce the background necessary to give the proofs of the above two theorems. In Section 8.2 and Section 8.3 we will prove Theorem 8.0.1 and Theorem 8.0.2. In Section 8.4 we will describe when Theorem 8.0.1 and Theorem 8.0.2 apply, and in Section 8.5 we will describe some techniques for partial realisation of the Casson-Sullivan invariant.

§ 8.1 | The surgery exact sequence

For a reference, see [Ran02, Chapter 13] in high-dimensions and for the 4-dimensional surgery exact sequence see [FQ90, Chapter 11] or [BKK+21, Chapter 22]. We will use the surgery sequence in both the smooth and topological categories, so to simplify the notation we will use CAT to stand in for both DIFF and TOP.

The surgery exact sequence consists of three types of objects, which we now recall: the structure set, the normal invariants, and the L-groups.

Definition 8.1.1. Let $(M, \partial M)$ be a connected CAT n-manifold with (potentially empty) boundary. Then the CAT-*structure set* of M, denoted as $\mathcal{S}_{\mathrm{CAT}}(M, \partial M)$ is the set of all equivalence classes of pairs (N, φ) where N is a CAT-n-manifold and φ a homotopy equivalence $\varphi\colon N \xrightarrow{\simeq} M$ such that the restriction $\varphi|_{\partial N}\colon \partial N \to \partial M$ is a CAT-isomorphism. The equivalence relation is that $(N, \varphi) \sim (N', \varphi')$ if there exists a (relative) CAT-h-cobordism $(W; N, N')$ with a homotopy equivalence

$$\Phi\colon W \to M \times [0,1]$$

such that $\Phi|_N = \varphi\colon N \to M \times \{0\}$ and $\Phi|_{N'} = \varphi'\colon N' \to M \times \{1\}$ and such that

$$\Phi|_{\partial N \times [0,1]} = (\varphi|_{\partial N} \times \mathrm{Id}_{[0,1]})\colon \partial N \times [0,1] \to \partial M \times [0,1].$$

Similarly, there is a *simple* CAT-*structure set*, denoted $\mathcal{S}^s_{\mathrm{CAT}}$, which is defined analogously to the regular structure set but with all homotopy equivalences replaced with simple homotopy equivalences (and hence all h-cobordisms replaced with s-cobordisms). Sometimes will we write $\mathcal{S}^h_{\mathrm{CAT}}$ to specify that we mean the regular structure set.

We now define the normal invariants.

Proposition 8.1.2. *Let $G(k)$ be the monoid of homotopy equivalences $S^{k-1} \to S^{k-1}$, let G denote the direct limit of the inclusions $G(k) \hookrightarrow G(k+1)$. Then there are fibration sequences*

$$G/\mathrm{O} \to \mathcal{B}\mathrm{O} \to \mathcal{B}\mathrm{G} \to \mathcal{B}(G/\mathrm{O})$$

and

$$G/\mathrm{TOP} \to \mathcal{B}\mathrm{TOP} \to \mathcal{B}\mathrm{G} \to \mathcal{B}(G/\mathrm{TOP}).$$

Proof. See [Ran02, Chapter 9] and [FQ90, Chapter 11]. Note that again G/O and G/TOP are defined as the homotopy fibres of $\mathrm{O} \to G$ and $\mathrm{TOP} \to G$, respectively, and that $\mathcal{B}(G/\mathrm{O})$, $\mathcal{B}(G/\mathrm{TOP})$ and the rightmost fibrations exist by Boardman-Vogt [BV68]. □

Definition 8.1.3. Let $(M, \partial M)$ be a connected n-manifold with (potentially empty) boundary ∂M which already has a CAT-structure. Then the *normal invariants* of M, denoted as $\mathcal{N}_{\mathrm{CAT}}(M, \partial M)$ is the set of homotopy classes of maps $[(M, \partial M), G/\mathrm{O}]$ if CAT = DIFF or $[(M, \partial M), G/\mathrm{TOP}]$ if CAT = TOP.

Remark 8.1.4. The above definition is one way of defining the normal invariants for a given manifold. An equivalent formulation is that a CAT normal invariant for a CAT-manifold $(M, \partial M)$ is a CAT-manifold $(N, \partial N)$ together with a so-called *degree one normal map* $f\colon N \to M$ which restricts to a CAT-isomorphism on ∂N. For more information, see [Ran02, Chapter 9].

The reader should recall the definition of the surgery obstruction groups (see Definition 6.1.1).

We can now state the surgery exact sequence, due to Browder, Novikov, Sullivan and Wall (and Freedman-Quinn in dimension 4).

Theorem 8.1.5 ([Wal70, Theorem 10.8], [FQ90, Theorem 11.3A])**.** *Let M be a compact, connected, oriented n-dimensional* CAT*-manifold. Then for* CAT = DIFF *and $n \geq 5$ we have the following exact sequence of pointed sets, which continues to the left in the obvious manner:*

$$\cdots \longrightarrow L_{n+2}(\mathbb{Z}[\pi]) \longrightarrow \mathcal{S}_{\mathrm{DIFF}}(M \times I, \partial) \longrightarrow [(M \times I, \partial), G/\mathrm{O}] \longrightarrow L_{n+1}(\mathbb{Z}[\pi])$$

$$\longrightarrow \mathcal{S}_{\mathrm{DIFF}}(M, \partial) \longrightarrow [(M, \partial), G/\mathrm{O}] \longrightarrow L_n(\mathbb{Z}[\pi]).$$

And for CAT = TOP *and $n \geq 5$ (and $n = 4$ provided that π is a good group; see Remark 1.2.7) we have the following analogous exact sequence of abelian groups:*

$$\cdots \longrightarrow L_{n+2}(\mathbb{Z}[\pi]) \longrightarrow \mathcal{S}_{\mathrm{TOP}}(M \times I, \partial) \longrightarrow [(M \times I, \partial), G/\mathrm{TOP}] \longrightarrow L_{n+1}(\mathbb{Z}[\pi])$$

$$\longrightarrow \mathcal{S}_{\mathrm{TOP}}(M, \partial) \longrightarrow [(M, \partial), G/\mathrm{TOP}] \longrightarrow L_n(\mathbb{Z}[\pi]).$$

Furthermore, there are simple versions of both of these exact sequences where the L-groups are replaced by the simple L-groups and the structure sets are replaced by the simple structure sets.

§ 8.2 | Forming mapping cylinders from the surgery exact sequence

Now we specialise to the case of interest. Let X be a compact, connected, smooth 4-manifold with boundary ∂X. The obstruction to lifting an element of $\mathcal{N}_{\mathrm{TOP}}$ to $\mathcal{N}_{\mathrm{DIFF}}$ is given by the map ξ_* induced by the fibration

$$G/\mathrm{O} \to G/\mathrm{TOP} \xrightarrow{\xi} \mathcal{B}(\mathrm{TOP}/\mathrm{O}).$$

We will consider the following augmented part of the TOP and DIFF surgery exact sequences for X:

$$\begin{array}{ccccc} & & [(X \times I, \partial), G/\,\mathrm{O}] & & \\ & & \big\downarrow & & \\ \mathcal{S}^s_{\mathrm{TOP}}(X \times I, \partial) & \longrightarrow & [(X \times I, \partial), G/\,\mathrm{TOP}] & \xrightarrow{\theta} & L^s_5(\mathbb{Z}[\pi]) \\ & & \big\downarrow \xi_* & & \\ & & [(X \times I, \partial), \mathcal{B}(\mathrm{TOP}/\mathrm{O})] & & \end{array}$$

The idea is to construct a mapping cylinder for a homeomorphism with non-trivial Casson-Sullivan invariant from this sequence. The way we will do this is by finding an element $N \in \mathcal{N}_{\mathrm{TOP}}(X \times I, \partial)$ which has vanishing surgery obstruction $\theta(N)$, but has $\xi_*(N) \neq 0$. First, we need to understand the map ξ_* more, which we do via the following two lemmas.

Lemma 8.2.1. *We have an isomorphism*

$$[(X \times I, \partial), G/\,\mathrm{TOP}] \cong H^2(X \times I, \partial; \mathbb{Z}/2) \oplus H^4(X \times I, \partial; \mathbb{Z}).$$

Proof. This follows from the work of Sullivan [Sul96], which can be found in [Ran02]. Instead though, we will refer to [MM79] for its exposition on this topic. In particular, it follows from [MM79, Remark 4.36] that

$$(G/\,\mathrm{TOP}[2])_6 \simeq K(\mathbb{Z}_{(2)}, 4) \times K(\mathbb{Z}/2, 2)$$

where $G/\,\mathrm{TOP}[2]$ denotes the 2-localisation of $G/\,\mathrm{TOP}$ and $(G/\,\mathrm{TOP})_6$ denotes its 6th Postnikov stage. From this, one can use [MM79, 4.35] to see that

$$(G/\,\mathrm{TOP})_6 \simeq K(\mathbb{Z}, 4) \times K(\mathbb{Z}/2, 2)$$

and, together with the fact that $X \times I$ is a 5-dimensional CW-complex, a standard obstruction theoretic argument completes the proof of the lemma. □

Lemma 8.2.2. *We have the following commutative diagram.*

$$\begin{array}{ccc} [(X \times I, \partial), G/\,\mathrm{TOP}] & \xrightarrow{\cong} & H^2(X \times I, \partial; \mathbb{Z}/2) \oplus H^4(X \times I, \partial; \mathbb{Z}) \\ \big\downarrow \xi_* & & \big\downarrow m \\ [(X \times I, \partial), \mathcal{B}(\mathrm{TOP}/\mathrm{O})] & \xrightarrow{\cong} & H^4(X \times I, \partial; \mathbb{Z}/2) \end{array}$$

where m is the map sending $(x, y) \mapsto (\mathrm{red}_2(y))$.

Proof. First, note that the bottom horizontal isomorphism is given by there being a 7-connected map from $\mathcal{B}(\mathrm{TOP}/\mathrm{O}) \to K(\mathbb{Z}/2, 4)$ (see Section 5.2), and the top horizontal isomorphism is given by Lemma 8.2.1.

Morita [Mor72, Proposition 3] gives us that m is the map sending $(x, y) \mapsto \mathrm{Sq}^2(x) + \mathrm{red}_2(y)$. Now it suffices to show that $\mathrm{Sq}^2(x) = 0$. We have the following commutative diagram (by the naturality of Steenrod squares):

$$\begin{array}{ccc} \widetilde{H}^2(X \times I, \partial; \mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & \widetilde{H}^4(X \times I, \partial; \mathbb{Z}/2) \\ \cong\uparrow & & \uparrow\cong \\ \widetilde{H}^1(X; \mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & \widetilde{H}^3(X; \mathbb{Z}/2) \end{array}$$

and the lower horizontal map must vanish since $\mathrm{Sq}^i\colon H^j(-;\mathbb{Z}/2) \to H^{j+i}(-;\mathbb{Z}/2)$ is the zero map for $i < j$. □

Now we construct the mapping cylinder.

Proposition 8.2.3. *Let X be a compact, connected, smooth 4-manifold with good fundamental group. Let $N \in \mathcal{N}_{\mathrm{TOP}}(X \times I, \partial)$ be an element of the normal invariants which has $\theta(N) = 0$. Then N can be lifted to an element of the structure set which is homeomorphic to a mapping cylinder $M_f = (X \times I) \cup_f X$ for some homeomorphism f.*

Proof. Assume that we have an element $N \in \mathcal{N}_{\mathrm{TOP}}(X \times I, \partial)$. By Remark 8.1.4 this means we can consider N to be a manifold $(N, \partial N)$ together with ∂N homeomorphic to $\partial(X \times I)$, and hence ∂N has an induced smooth structure given by X. Further we assume that $\theta(N) = 0$. This means that we can lift N to an element (also denoted by N) in the structure set $\mathcal{S}^s_{\mathrm{TOP}}(X \times I, \partial)$. The homeomorphism $\partial N \approx \partial(X \times I) \cong X \times \{0\} \cup \partial X \times I \cup X \times \{1\}$ induces a decomposition $\partial N = \partial_+ N \cup \partial_0 N \cup \partial_- N$. If we define $D := \partial(\partial_+ N)$, then we may assume that $\partial_0 N \cong D \times I$. By the relative s-cobordism theorem [FQ90, Theorem 7.1A] (which applies since X has good fundamental group by assumption) there exists a homeomorphism relative to $\partial_+ N \cup (D \times I)$

$$(N, \partial_+ N) \xrightarrow{\approx} (\partial_+ N \times I, \partial_+ N \times \{0\})$$

such that this homeomorphism restricts to the identity on $\partial_+ N \cup (D \times I)$. Let $\widetilde{f}$ denote the restriction of this homeomorphism to $\partial_- N \to \partial_+ N \times \{1\}$. Since $N \in \mathcal{S}^s_{\mathrm{TOP}}(X \times I, \partial)$, we also have a (simple) homotopy equivalence, restricting to a homeomorphism on the boundary

$$(N; \partial_+ N, \partial_0 N, \partial_- N) \xrightarrow{\simeq} (X \times I; X \times \{0\}, \partial X \times I, X \times \{1\})$$

Putting this together, we have the commutative diagram

$$\begin{array}{ccc} \partial_- N & \xrightarrow{\widetilde{f}} & \partial_+ N \\ \downarrow\approx & & \downarrow\approx \\ X & \xrightarrow{f} & X \end{array}$$

where f is defined such that the diagram commutes.

It follows that our constructed element $N \in \mathcal{S}^s_{\mathrm{TOP}}(X \times I, \partial)$ is homeomorphic to the mapping cylinder $M_f := (X \times I) \cup_f X$, restricting to a diffeomorphism on the boundary by construction. □

§ 8.3 | Proof of the unstable realisation theorem

We now define the technical "realisability condition" which was mentioned in the introduction.

Definition 8.3.1. Let X be a closed, connected, smooth, orientable 4-manifold with $\pi_1(X) \cong \pi$ where π is a good group. We say that X satisfies the *Casson-Sullivan realisability condition* if the surgery obstruction map (after reidentifying the normal invariants using Lemma 8.2.1)

$$\theta\colon H^2(X \times I, \partial; \mathbb{Z}/2) \oplus H^4(X \times I, \partial; \mathbb{Z}) \to L_5(\mathbb{Z}[\pi])$$

is such that for every $y \in H^4(X \times I, \partial; \mathbb{Z})$ there exists an $x \in H^2(X \times I, \partial; \mathbb{Z}/2)$ such that $\theta(x, y) = 0$.

We begin with a simple but essential observation.

Lemma 8.3.2. *Let N be an element as in Proposition 8.2.3 and let M_f be the associated mapping cylinder. Then $\mathrm{cs}(f) = \varpi^{-1}\xi_*(N)$.*

Proof. Note that $\xi_*(N)$ is the obstruction to lifting N to an element of the smooth structure set $\mathcal{S}^s_{\mathrm{DIFF}}(X \times I, \partial)$ and hence $\xi_*(N) = \mathrm{ks}(M_f)$. The lemma follows by the definition of the Casson-Sullivan invariant (Definition 5.2.9). □

Now it only remains to prove the main theorem, but most of the work has already been done.

Proof of Theorem 8.0.1. Let X be as in the statement of the theorem and let $\eta \in H^3(X; \mathbb{Z}/2)$.

Recall the diagram from Lemma 8.2.2. We claim that for any $z \in H^4(X \times I, \partial; \mathbb{Z}/2)$ there exists a $y \in H^4(X \times I, \partial; \mathbb{Z})$ such that $m(0, y) = \mathrm{red}_2(y) = z$. By the universal coefficients theorem, we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1(H_3(X \times I, \partial), \mathbb{Z}) & \longrightarrow & H^4(X \times I, \partial; \mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_4(X \times I, \partial), \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{red}_2} & & \downarrow{\scriptstyle \mathrm{red}_2} & & \downarrow{\scriptstyle \mathrm{red}_2} & & \\
0 & \longrightarrow & \mathrm{Ext}^1(H_3(X \times I, \partial), \mathbb{Z}/2) & \longrightarrow & H^4(X \times I, \partial; \mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}(H_4(X \times I, \partial), \mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

where the rows are short exact sequences and the vertical maps are given by reduction of coefficients. Both the leftmost and rightmost vertical maps are surjective, and hence, by the five-lemma, the middle vertical map is surjective. This completes the proof of the claim.

So, given any $\eta \in H^3(X, \partial X; \mathbb{Z}/2)$, we can find an element $y \in H^4(X \times I, \partial; \mathbb{Z})$ such that

$$\varpi^{-1} m(0, y) = \eta.$$

By the realisability condition, there exists an $x \in H^2(X \times I, \partial; \mathbb{Z}/2)$ such that $\theta(x, y) = 0$. Define N_η such that N_η maps to (x, y) under the isomorphism given in Lemma 8.2.1. By Lemma 8.2.2, it follows that $\xi_*(N_\eta) = \varpi\eta$. Hence, by Proposition 8.2.3 and Lemma 8.3.2, we have that there exists a homeomorphism $f \colon X \to X$ with $\mathrm{cs}(f) = \eta$. □

We finish this section by proving Theorem 8.0.2, which follows easily from the construction of the non-pseudo-smoothable homeomorphisms produced by Theorem 8.0.1.

Proof of Theorem 8.0.2. Let X be as in the statement of the theorem. Then, as in the above proof of Theorem 8.0.1, for every non-zero $\eta \in H^3(X; \mathbb{Z}/2)$ there exists a mapping cylinder $M_f = (X \times I) \cup_f X$ such that $\mathrm{cs}(f) = \eta \neq 0$. Hence f is not stably pseudo-isotopic to a diffeomorphism by Proposition 5.4.1 and Proposition 5.4.4. However, now also note that $M_f \in \mathcal{S}^s_{\mathrm{TOP}}(X \times I, \partial)$ and hence we have a homotopy equivalence

$$((X \times I) \cup_f X; X \times \{0\}, X \times \{1\}) \simeq (X \times I; X \times \{0\}, X \times \{1\})$$

which by construction restricts to the identity Id_X on $X \times \{0\}$ and f on $X \times \{1\}$. By post-composing this homotopy equivalence with the projection to X, this produces a homotopy between f and Id_X. □

§ 8.4 | Applications

In light of the previous section, it is natural to ask for which 4-manifolds do the results Theorem 8.0.1 and Theorem 8.0.2 apply. This section is devoted to giving such examples. In all that follows, let X be a closed, connected, smooth, orientable 4-manifold with $\pi_1(X) \cong \pi$.

Let Γ be a finite cyclic group. Then it follows from Bak [Bak76] that $L^s_5(\mathbb{Z}[\Gamma]) = 0$. Hence, for $\pi \cong \Gamma$ we have that the surgery obstruction vanishes trivially and hence Theorem 8.0.1 and Theorem 8.0.2 apply. We can say more, however.

Proposition 8.4.1. *Let X be a compact, connected, smooth, orientable 4-manifold with $\pi_1(X) \cong \Gamma$ with Γ a finite group such that $SK_1(\mathbb{Z}[\Gamma]) = 0$ or such that $SK_1(\mathbb{Z}[\rho]) = 0$ where ρ denotes the Sylow 2-subgroup of Γ. Then Theorem 8.0.1 and Theorem 8.0.2 both apply to X.*

Remark 8.4.2. For a group ring $\mathbb{Z}[\Gamma]$ we define $SK_1(\mathbb{Z}[\Gamma])$ to be a certain subgroup of the algebraic K-theory group $K_1(\mathbb{Z}[\Gamma])$. More specifically, it is defined as the kernel of the inclusion induced map $K_1(\mathbb{Z}[\Gamma]) \to K_1(\mathbb{Q}[\Gamma])$. If Γ is abelian, then this is equivalently defined as the kernel of the determinant map $\det \colon K_1(\mathbb{Z}[\Gamma]) \to \mathbb{Z}[\Gamma]^\times$.

Proposition 8.4.1 follows from a result of Hambleton-Milgram-Taylor-Williams, which we now restate the relevant part of (adapted to our situation).

Theorem 8.4.3 ([HMTW88, Theorem A]). *Let M be a 5-dimensional manifold with boundary ∂M such that $\pi_1(M) \cong \pi$ is finite and $\operatorname{im}(SK_1(\mathbb{Z}[\rho]) \to SK_1(\mathbb{Z}[\pi])) = 0$, where ρ is the Sylow 2-subgroup of π. Then the surgery obstruction map*

$$\theta\colon [(M, \partial M), G/\mathrm{TOP}] \to L_5^s(\mathbb{Z}[\pi])$$

is given by

$$x \mapsto \kappa_3^s(c_*(\mathrm{Arf}_3(x))).$$

In the above theorem, κ_3^s denotes the map (constructed in [HMTW88, §1])

$$\kappa_3^s\colon H_3(\mathcal{B}\pi; \mathbb{Z}/2) \to L_5^s(\mathbb{Z}[\pi]),$$

where $c\colon M \to \mathcal{B}\pi$ is defined as the classifying map for the universal cover of M, and Arf denotes the Arf invariant.

Before we prove Proposition 8.4.1, we state a few groups Γ for which $SK_1(\Gamma)$ vanishes. This is the work of many mathematicians, and we direct the reader to [Oli88, p.3-4] and the citations within for references.

(i) $\Gamma = \mathbb{Z}/2n$

(ii) $\Gamma = \mathbb{Z}/2^n \times \mathbb{Z}/2$

(iii) $\Gamma = (\mathbb{Z}/2)^n$.

(iv) $\Gamma = D_{2n}$, the dihedral group of order $4n$.

So Proposition 8.4.1 applies to any of the above groups. It also follows that Theorem 8.0.1 and Theorem 8.0.2 also apply to any Γ that has any of the above groups as its Sylow 2-subgroup. For example, let ρ be isomorphic to one of the above groups and let O be an odd-order group. Then $\Gamma := \rho \times O$ has ρ as its Sylow 2-subgroup and hence Proposition 8.4.1 also applies to Γ.

Proof of Proposition 8.4.1. Let X satisfy the hypotheses of Proposition 8.4.1 and let $b \in H^4(X \times I; \mathbb{Z})$. We need to show that there exists an $a \in H^2(X \times I; \mathbb{Z}/2)$ such that $\theta(a, b) = 0$. We claim that we can simply take $a = 0$ regardless of b.

Note that $X \times I$ is a 5-dimensional manifold with boundary and by assumption we have that $\operatorname{im}(SK_1(\mathbb{Z}[\rho]) \to SK_1(\mathbb{Z}[\Gamma])) = 0$, hence Theorem 8.4.3 applies to $X \times I$. Theorem 8.4.3 then tells us that the surgery obstruction map θ factors as

$$[(X \times I, \partial), G/\mathrm{TOP}] \to H^2(X \times I, \partial; \mathbb{Z}/2) \cong H_3(X; \mathbb{Z}/2) \to H_3(\mathcal{B}\pi; \mathbb{Z}/2) \xrightarrow{\kappa_3^s} L_5^s(\mathbb{Z}[\pi]),$$

where the first map is the projection, using Lemma 8.2.1. Hence, $\theta(a, b)$ does not depend on b, and so for all $b \in H^4(X \times I, \partial; \mathbb{Z}/2)$ we have that $\theta(0, b) = \theta(0, 0) = 0$. □

§ 8.5 | Partial unstable realisation of the Casson-Sullivan invariant

The purpose of this section is to find examples where we can partially realise the Casson-Sullivan invariant using the method described in Section 8.2. The way we will do this is by considering the assembly maps for the surgery obstruction map θ and comparing them using spectral sequences to surgery obstruction maps that we have full realisation for. In all that follows, let X be a closed, connected, smooth, orientable 4-manifold with $\pi_1(X) \cong \pi$.

First we give the relevant notation. Let $\varepsilon \in \{-\infty\} \cup \{\dots, -1, 0, 1, 2\}$ be the decoration, and recall that $\varepsilon = 2$ refers to $\varepsilon = s$ and $\varepsilon = 1$ refers to $\varepsilon = h$. For more information about decorations, see [Lüc23]. Let $\mathbb{L}^\varepsilon_\bullet(R)$ denote the quadratic L-theory spectrum of a ring R with decoration ε (this is a spectrum that has homotopy groups $\pi_k(\mathbb{L}^\varepsilon_\bullet(R)) = L^\varepsilon_k(R))$ and let $\mathbb{L}_\bullet := \mathbb{L}_\bullet(\mathbb{Z})$ (note that, as suggested by the notation, this is independent of our choice of decoration). We will use $\mathbb{L}_\bullet\langle 1\rangle$ to denote the 1-connective cover of $\mathbb{L}_\bullet$ (a spectrum which has $\pi_k(\mathbb{L}_\bullet\langle 1\rangle) = 0$ for all $k < 1$ and has $\pi_k(\mathbb{L}_\bullet\langle 1\rangle) = \pi_k(\mathbb{L}_\bullet)$ for $k \geq 1$). Then these spectra determine a generalised (co)homology theory and we have the following factorisation of the surgery obstruction map θ^ε (note that the notation used will be explained below).

$$\begin{array}{ccc}
\mathcal{N}_{\mathrm{TOP}}(X \times I, \partial) & \xrightarrow{\theta^\varepsilon} & L^\varepsilon_5(\mathbb{Z}[\pi]) \\
{\scriptstyle a}\downarrow\cong & & \cong\uparrow{\scriptstyle g} \\
H^0(X \times I, \partial; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & & H^\pi_5(*_{\mathcal{ALL}}; \mathbb{L}^\varepsilon_\bullet) \\
{\scriptstyle b}\downarrow\cong & & \uparrow{\scriptstyle \sigma^\varepsilon} \\
H_5(X \times I; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & & H^\pi_5(*_{\mathcal{TRIV}}; \mathbb{L}^\varepsilon_\bullet) \\
{\scriptstyle c}\downarrow & & \cong\uparrow{\scriptstyle e} \\
H_5(\mathcal{B}\pi; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & \xrightarrow[d]{} & H_5(\mathcal{B}\pi; \mathbb{L}^\varepsilon_\bullet)
\end{array}$$

The isomorphism a arises via the definition of cohomology with coefficients in a spectrum. Since we have that $(\mathbb{L}_\bullet\langle 1\rangle)_0 \simeq G/\mathrm{TOP}$ we have that

$$H^0(X \times I, \partial; \mathbb{L}_\bullet) \cong [X \times I, \partial; G/\mathrm{TOP}] \cong \mathcal{N}_{\mathrm{TOP}}(X \times I, \partial).$$

The isomorphism b is given by Sullivan-Ranicki duality. To define the map c, we factor it as the composition of maps given in the diagram below where $\widetilde{X \times I}$ denotes the universal cover of $X \times I$.

$$\begin{array}{ccccc}
H_5(X \times I; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & \xrightarrow{\cong} & H^\pi_5(\widetilde{X \times I}; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & \xrightarrow{\cong} & H_5(\mathcal{E}\pi \times_\pi (\widetilde{X \times I}); \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) \\
{\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle c'} \\
H_5(\mathcal{B}\pi; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & \xleftarrow[\cong]{} & H_5(\mathcal{E}\pi \times_\pi \{\mathrm{pt}\}; \mathbb{L}^\varepsilon_\bullet\langle 1\rangle) & \xleftarrow{} &
\end{array}$$

To define the map d, recall that we have a map $\mathbb{L}_\bullet^\varepsilon\langle 1\rangle \to \mathbb{L}_\bullet^\varepsilon$ by the definition of a 1-connective cover. Then d is the induced map on homology via this map. The definition of the isomorphism e follows from the definition of the π-equivariant homology of the Orb(π)-space $*_{\mathcal{TRIV}}$ with coefficients in a spectrum. For details on this, see [DL98, Ex 5.5]. The isomorphism g is again given by the definition of the π-equivariant homology. The map σ^ε is what we will call the *ε-assembly map*. Since we need to use the s-cobordism theorem to construct our mapping cylinders, we will need to consider the case $\varepsilon = s$, for which this assembly map is certainly not an isomorphism in general. Composing all of these maps together we get a factorisation of θ^s.

Now we want to use this formalism to try to understand θ^s for some groups for which we cannot use Proposition 8.4.1. Let Γ be a finite cyclic group such that we have a non-trivial homomorphism $h\colon \Gamma \to \pi$. In what follows we will condense the 'assembly map' by setting $\overline{\sigma} := g \circ \sigma^s \circ e$.

Consider the following diagram.

$$\begin{array}{ccccc}
\mathcal{N}_{\mathrm{TOP}}(X \times I, \partial) & \xrightarrow{\theta^s} & L_5^s(\mathbb{Z}[\pi]) & \longleftarrow & L_5^s(\mathbb{Z}[\Gamma]) = 0 \\
{\scriptstyle a}\downarrow{\scriptstyle \cong} & & {\scriptstyle \overline{\sigma}}\uparrow & & {\scriptstyle \overline{\sigma}}\uparrow \\
H_5(X \times I; \mathbb{L}_\bullet^s\langle 1\rangle) & \longrightarrow & H_5(\mathcal{B}\pi; \mathbb{L}_\bullet^s) & \xleftarrow[h_*]{} & H_5(\mathcal{B}\Gamma; \mathbb{L}_\bullet^s)
\end{array}$$

By naturality of the assembly maps, if a normal invariant is hit by the map h_* after being mapped down to $H_5(\mathcal{B}\pi; \mathbb{L}_\bullet^s)$, then its surgery obstruction must be zero. It follows that if the map h_* is surjective then θ^s is the zero map. To try to understand this map h_* better we will use the Atiyah-Hirzebruch spectral sequence (AHSS). Recall that the AHSS is a homology spectral sequence that computes the generalised homology of a space in terms of the regular homology of that space with coefficients in the generalised homologies of a point. In our particular case it computes, for a space K,

$$E^2_{p,q} = H_p(K; \pi_q(\mathbb{L}_\bullet\langle 1\rangle)) \implies E^\infty_{p,q} = H_{p+q}(K; \mathbb{L}_\bullet\langle 1\rangle).$$

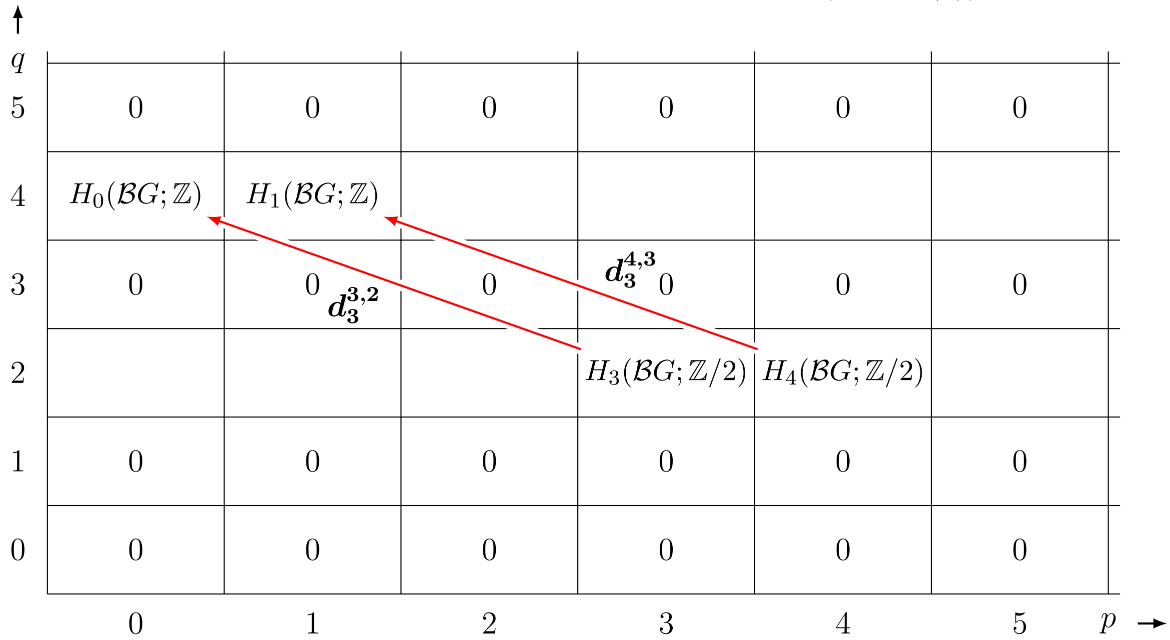
Recall that $\pi_q(\mathbb{L}_\bullet\langle 1\rangle) = L_q(\mathbb{Z})$ for $q > 0$ and is zero otherwise. We first wish to compute $H_5(\mathcal{B}G; \mathbb{L}_\bullet\langle 1\rangle)$ for the cases $G = \pi$ and $G = \Gamma$. We show the relevant terms below of the E^3-page, along with the relevant third-page differentials, in Figure 8.1.

The conclusions we can draw from this spectral sequence are contained in the following lemma.

Lemma 8.5.1. *Let Γ be a finite cyclic group of order $2n$, let $\pi \cong \mathbb{Z} \times \Gamma$ and let d_G denote the differential $d_3^{4,3}$ denoted above. Then we have the following commutative diagram where the rows are exact.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker} d_\Gamma & \longrightarrow & H_5(\mathcal{B}\Gamma; \mathbb{L}_\bullet\langle 1\rangle) & \longrightarrow & H_3(\mathcal{B}\Gamma; \mathbb{Z}/2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{coker} d_\pi & \longrightarrow & H_5(\mathcal{B}\pi; \mathbb{L}_\bullet\langle 1\rangle) & \longrightarrow & H_3(\mathcal{B}\pi; \mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

Figure 8.1: The E^3-page of the spectral sequence for computing $H_*(\mathcal{B}G; \mathbb{L}_\bullet\langle 1\rangle)$ with the relevant terms and differentials shown for computing $H_5(\mathcal{B}G; \mathbb{L}_\bullet\langle 1\rangle)$.



and $H_3(\mathcal{B}\Gamma; \mathbb{Z}/2) \cong \mathbb{Z}/2$*,* $\operatorname{coker} d_\Gamma \cong \mathbb{Z}/2n$ *or* $\mathbb{Z}/n$*,* $H_3(\mathcal{B}\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$*, and* $\operatorname{coker} d_\pi \cong \mathbb{Z} \oplus \mathbb{Z}/2n$ *or* $\mathbb{Z} \oplus \mathbb{Z}/n$*.*

Proof. First note that for any group G the differential $d_3^{3,2}$ vanishes since it is map from a torsion group to a torsion-free group. Then recall that a model for $\mathcal{B}\Gamma$ is given by the infinite lens space $L(2n)$, and that a model for $\mathcal{B}\pi$ is given by the space $S^1 \times L(2n)$. We have that $d_\Gamma\colon \mathbb{Z}/2 \to \mathbb{Z}/2n$ is either the zero map or is injective. Similarly, we get that $d_\pi\colon \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2n$ is either the zero map or the map $(a, b) \mapsto (0, nb)$ (the other possibilities may be ruled out by further comparing with the spectral sequence for $G = \mathbb{Z}$ and using naturality). From this we can conclude that the isomorphism types of the cokernels are as described in the lemma.

Finally, the diagonal $p + q = 5$ computes the associated graded for $H_5(\mathcal{B}G; \mathbb{L}_\bullet\langle 1\rangle)$ and hence we have the short exact sequences which form the rows of the stated commutative diagram. The vertical maps are then induced by the obvious inclusion map $\Gamma \to \pi$ and the diagram commutes by naturality of spectral sequences. This completes the proof. □

We have that $H^4(X \times I, \partial; \mathbb{Z}/2) \cong H^3(X; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the aim is now to realise the element $(0, 1)$ as the Casson-Sullivan invariant of a homeomorphism $f\colon X \to X$. Following the realisation procedure laid out in Section 8.2 and Section 8.3, by Proposition 8.2.3 and Lemma 8.3.2 it suffices to show that there exists an element $y \in \mathcal{N}_{\mathrm{TOP}}(X \times I, \partial) \cong [(X \times I, \partial), G/\mathrm{TOP}]$ such that the map m from Lemma 8.2.2 sends y to $(0, 1) \in H^4(X \times I, \partial; \mathbb{Z}/2)$ and such that $\theta(y) = 0$.

Lemma 8.5.2. *Let* Γ *be a finite cyclic group of order* $2n$*,* $\pi = \mathbb{Z} \times \Gamma$ *and let* $\iota\colon \Gamma \to \pi$ *be the obvious inclusion map. The leftmost vertical map in the commutative diagram* *from Lemma 8.5.1 is then ι_* and is given by*

$$\iota_* \colon \begin{cases} \mathbb{Z}/2n \\ \mathbb{Z}/n \end{cases} \to \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2n \\ \mathbb{Z} \oplus \mathbb{Z}/n \end{cases}$$
$$a \mapsto (0, a).$$

Proof. We see this from the following commutative diagram, where we have identified $H_1(\mathcal{B}G; \mathbb{Z}) \cong \mathrm{ab}(G)$ for any group G, where $\mathrm{ab}(G)$ denotes the abelianisation of G.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\cong} & \mathrm{ab}(\Gamma) & \longrightarrow & \operatorname{coker} d_\Gamma \\ \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow{\scriptstyle \iota_*} \\ \pi & \xrightarrow{\cong} & \mathrm{ab}(\pi) & \longrightarrow & \operatorname{coker} d_\pi \end{array}$$

Since $\iota(a) = (0, a)$, the commutativity of the above diagram gives the result regardless of the isomorphism types of the cokernels given in Lemma 8.5.1. $\square$

Proposition 8.5.3. *Let X be a closed, connected, smooth, orientable 4-manifold with $\pi_1(X) \cong \pi = \mathbb{Z} \times \Gamma$. Then $H^3(X; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and there exists a self-homeomorphism $f \colon X \to X$ such that $\mathrm{cs}(f) = (0, 1) \in H^3(X; \mathbb{Z}/2)$.*

Proof. First, note that we can similarly use the AHSS to compute $H_5(X \times I; \mathbb{L}_\bullet\langle 1 \rangle)$. Of course, we already know the answer since $H_5(X \times I; \mathbb{L}_\bullet\langle 1 \rangle) \cong H_1(X; \mathbb{Z}) \oplus H_3(X; \mathbb{Z}/2)$ by Lemma 8.2.1, but this means that we can fit this data into a larger commutative diagram with the one from Lemma 8.5.1, which we do now below.

$$\begin{array}{ccccccc} \operatorname{coker} d_\Gamma & \rightarrowtail & H_5(\mathcal{B}\Gamma; \mathbb{L}_\bullet\langle 1 \rangle) & & \twoheadrightarrow & & H_3(\mathcal{B}\Gamma; \mathbb{Z}/2) \\ & & & \searrow{\scriptstyle \theta^s} & & & \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & L_5^s(\mathbb{Z}[\Gamma]) & & \downarrow{\scriptstyle \iota_*} \\ & & & & \downarrow & & \\ \operatorname{coker} d_\pi & \rightarrowtail & H_5(\mathcal{B}\pi; \mathbb{L}_\bullet\langle 1 \rangle) & & \twoheadrightarrow & & H_3(\mathcal{B}\pi; \mathbb{Z}/2) \\ & & & \searrow{\scriptstyle \theta^s} & & & \\ \uparrow & & \uparrow{\scriptstyle a} & & L_5^s(\mathbb{Z}[\pi]) & & \uparrow \\ H_1(X; \mathbb{Z}) & \rightarrowtail & H_5(X \times I; \mathbb{L}_\bullet\langle 1 \rangle) & & \twoheadrightarrow & & H_3(X; \mathbb{Z}/2) \end{array}$$

Note that we have omitted the zeroes from the ends of the rows, but the rows are still split exact sequences. Let $x \in H_5(X \times I; \mathbb{L}_\bullet\langle 1 \rangle)$ be the element corresponding to

$$((0, 1), 0) \in H^4(X \times I, \partial; \mathbb{Z}) \oplus H^2(X \times I, \partial; \mathbb{Z}/2)$$

via the isomorphism from Lemma 8.2.1. If we can show that the surgery obstruction $\theta^s(x) = 0$, then by Proposition 8.2.3 we can construct a homeomorphism $f \colon X \to X$ which has $\mathrm{cs}(f) = m((0, 1), 0) = (0, 1)$.

Using the fact that x is hit by an element from $H_1(X;\mathbb{Z})$ and the description of the leftmost map denoted by ι_* from Lemma 8.5.2, a simple diagram chase tells us that $a(x)$ is in the image of ι_*. Hence by naturality of assembly maps, and the fact that $L_5^s(\mathbb{Z}[\Gamma]) = 0$, we have that $\theta(x) = 0$, completing the proof. □

Proposition 8.5.3 tells us that we can also find examples of homeomorphisms that are not stably (pseudo-)smoothable but are homotopic to the identity for 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2n$. Note also that the above arguments would have worked just as well with $\mathbb{Z}$ replaced with $\mathbb{Z}^n$, so we can also find non-smoothable homeomorphisms in those cases as well.

Chapter 9

Pseudo-isotopy of 3-manifolds

The aim of this chapter is to understand the difference between the smooth and topological categories for 3-manifolds. Let Y be a closed, oriented, smooth 3-manifold. We recall from the introduction the result of Cerf [Cer59, Hat83] which says that the natural map[1]

$$\mathrm{Diff}(Y) \to \mathrm{Homeo}(Y)$$

is a homotopy equivalence. A consequence of this result is that topological isotopies and smooth isotopies are "the same" for 3-manifolds. We wish to study whether the same holds also for pseudo-isotopies. To this end, we will look at spaces which are closely related to the homeomorphism and diffeomorphism groups, the so-called block homeomorphism and block diffeomorphism groups $\widetilde{\mathrm{Homeo}}(Y)$ and $\widetilde{\mathrm{Diff}}(Y)$, which we will define later. These will be spaces such that connected components correspond to the topological pseudo-mapping class group and smooth pseudo-mapping class group, respectively. The existence of these spaces were previously referred to in Remark 2.2.3.

In this chapter, we will prove Theorem 1.2.13 and its corollary Corollary 1.2.14, as well as the related Theorem 1.2.15. We now recall these.

Theorem 9.0.1. *Let Y be a closed, elliptic 3-manifold such that $H_1(Y;\mathbb{Z}/2)$ is not trivial. Then the natural map*

$$\widetilde{\mathrm{Diff}}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$$

is not 1-connected. In particular, it is not surjective on π_1.

This was Theorem 1.2.13. We recall its corollary, which was Corollary 1.2.14 from the introduction.

Corollary 9.0.2. *Let Y be as in Theorem 9.0.1. Then the natural map*

$$\mathrm{Homeo}(Y) \to \widetilde{\mathrm{Homeo}}(Y)$$

is not 1-connected. In particular, it is not surjective on π_1.

[1]Whilst this map is an inclusion of sets, it is not an inclusion of spaces, since the topology on $\mathrm{Diff}(Y)$ is finer than the topology on $\mathrm{Homeo}(Y)$ restricted to the subset of diffeomorphisms. Cerf calls such a pair of spaces "une paire topologique".

Remark 9.0.3. The classes of 3-manifolds that Theorem 9.0.1 and Corollary 9.0.2 apply to are the following.

(i) Lens spaces with even order fundamental group.

(ii) Metacyclic prism manifolds.

(iii) Octahedral manifolds.

See the proof of Proposition 9.3.2 for the different types of elliptic 3-manifolds and their descriptions. Metacyclic prism manifolds are additionally defined in Definition 9.2.4.

Finally, as an application of the work in this chapter to 4-manifolds, we will prove the following theorem, which was Theorem 1.2.15 from the introduction.

Theorem 9.0.4. *There exists a* 4*-manifold* X *with boundary an elliptic* 3*-manifold such that there exists a self-homeomorphism* $f\colon X \to X$ *which is not pseudo-smoothable but is absolutely pseudo-smoothable.*

This theorem demonstrates that Proposition 4.4.2 does not hold if we replace isotopy with pseudo-isotopy. It is a corollary of Theorem 9.0.1 that the proof method for Proposition 4.4.2 fails, since Theorem 9.0.1 implies that there exist non-smoothable "pseudo-loops" of homeomorphisms of 3-manifolds. A specific construction, and our work on the Casson-Sullivan invariant form Chapter 5 is required though to actually show that it fails, and hence prove Theorem 9.0.4.

§ 9.0.1 | Chapter outline

We start with Section 9.1, where we define and introduce the block diffeomorphism and block homeomorphism groups. In Section 9.2, we survey what is known about pseudo-mapping class groups of 3-manifolds. We then prove Theorem 9.0.1 and Corollary 9.0.2 in Section 9.3. Finally, we finish this chapter (and this thesis) by proving Theorem 9.0.4 in Section 9.4.

§ 9.1 | Block diffeomorphisms and block homeomorphisms

Recall the definition of the homeomorphism group and diffeomorphism group from Definition 2.2.2.

We define the block diffeomorphism group and block homeomorphism group via semi-simplicial spaces.

Definition 9.1.1. Let M be a (smooth) manifold with (potentially empty) boundary ∂M and let Δ^p denote the standard p-simplex with face maps $s_i\colon \Delta^p \to \Delta^{p-1}$ for $i = 0, \ldots, p$ and inclusions $r_i\colon \Delta^{p-1} \to \Delta^p$ for $i = 0, \ldots, p$. Let $\widetilde{\mathrm{Homeo}}(M, \partial M)_\bullet$ be the simplicial space defined as follows. We define $\widetilde{\mathrm{Homeo}}(M,\partial M)_0 := \mathrm{Homeo}(M,\partial M)$ as a space. Then we define the space of p-simplices inductively as

$$\begin{aligned}\widetilde{\mathrm{Homeo}}(M,\partial M)_p := \{ & f\colon M\times\Delta^p \xrightarrow{\approx} M\times\Delta^p \mid \\ & (f|_{M\times s_i(\Delta^p)}\colon M\times\Delta^{p-1}\xrightarrow{\approx} M\times\Delta^{p-1}) \in \widetilde{\mathrm{Homeo}}_\partial(M)_{p-1} \text{ for } i=0,\dots,p\},\end{aligned}$$

where the topology is given by the compact-open topology. Then the simplicial space structure on $\cup_{p=0,1,\dots}\widetilde{\mathrm{Homeo}}(M,\partial M)_p$ is given by the face maps

$$\begin{aligned}\widetilde{s}_i\colon \widetilde{\mathrm{Homeo}}(M,\partial M)_p &\to \widetilde{\mathrm{Homeo}}(M,\partial M)_{p-1}\\ f &\mapsto f|_{M\times s_i(\Delta^p)},\end{aligned}$$

for $i=0,\dots,p$, and the degeneracy maps

$$\widetilde{r}_i\colon \widetilde{\mathrm{Homeo}}(M,\partial M)_{p-1} \to \widetilde{\mathrm{Homeo}}(M,\partial M)_p$$

for $i=0,\dots,p$. The degeneracy maps $\widetilde{r}_i$ are defined as follows. Fix a face $r_i(\Delta^{p-1})$ of Δ^p. Then write

$$\Delta^p = \frac{r_i(\Delta^{p-1})\times I}{(0,t_1)\sim(0,t_2)},$$

where "0" means the image of the first vertex of Δ^{p-1} (with the standard ordering) under the map r_i. Now, using this identification, we define

$$\widetilde{r}_i(f)(m,(x,t)) := (\mathrm{proj}_M(f(m,x)), (\mathrm{proj}_{\Delta^p}(f(m,x)),t))$$

where $\mathrm{proj}_M\colon M\times\Delta^{p-1}\to M$ and $\mathrm{proj}_{\Delta^{p-1}}\colon M\times\Delta^{p-1}\to\Delta^{p-1}$ are the standard projection maps.

We then define the *block homeomorphism group of M* to be the topological group

$$\widetilde{\mathrm{Homeo}}(M,\partial M) := ||\widetilde{\mathrm{Homeo}}(M,\partial M)_\bullet|| = \left(\sqcup_p \widetilde{\mathrm{Homeo}}(M,\partial M)_p \times\Delta^p\right)/\sim$$

i.e. the geometric realisation of the simplicial space we have just built, where the equivalence relation $\sim$ is given by the face maps and degeneracy maps.

We similarly define the simplicial space $\widetilde{\mathrm{Diff}}(M,\partial M)_\bullet$ and the *block diffeomorphism group of M* as the geometric realisation

$$\widetilde{\mathrm{Diff}}(M,\partial M) := ||\widetilde{\mathrm{Diff}}(M,\partial M)_\bullet||.$$

Again (as in Definition 2.2.2), there are orientation preserving variants $\widetilde{\mathrm{Homeo}}^+(M,\partial M)$, $\widetilde{\mathrm{Diff}}^+(M,\partial M)$, and if $\partial M=\emptyset$ we will omit the boundary from the notation.

Remark 9.1.2. The group structure on $\widetilde{\mathrm{Homeo}}(M,\partial M)$ is defined as follows. Any pair of points in $\widetilde{\mathrm{Homeo}}(M,\partial M)$ can be represented by a pair of homeomorphisms $f\colon M\times\Delta^p\to M\times\Delta^p$ and $g\colon M\times\Delta^q\to M\times\Delta^q$ for some $p,q\geq 0$. Further, we may assume that $p=q$ by using the degeneracy maps if necessary. Then define $f\cdot g := f\circ g$.

The group structure on $\widetilde{\mathrm{Diff}}(M, \partial M)$ is defined analogously.

Lemma 9.1.3. *Let M be a (smooth) manifold. Then the maps given by the inclusions*

(i) $\mathrm{Homeo}(M, \partial M) \to \widetilde{\mathrm{Homeo}}(M, \partial M)$,

(ii) $\mathrm{Diff}(M, \partial M) \to \widetilde{\mathrm{Diff}}(M, \partial M)$,

(iii) $\widetilde{\mathrm{Diff}}(M, \partial M) \to \widetilde{\mathrm{Homeo}}(M, \partial M)$

are all continuous.

Proof. We only prove (i); the proofs for (ii) and (iii) are similar. Let $\widetilde{\iota}$ denote the inclusion map in question. It suffices to show that the preimage under $\widetilde{\iota}$ of any sub-basis element of $\widetilde{\mathrm{Homeo}}(M, \partial M)$ is open in $\mathrm{Homeo}(M, \partial M)$. We start by describing a sub-basis.

Since $\mathrm{Homeo}(M, \partial M)$ has the compact-open topology, a sub-basis is given by $C_M(K, U)$ for all $K \subset M$ compact and all $U \subset M$ open, where $C_M(K, U)$ consists of all homeomorphisms which map K into U. By the definition of $\widetilde{\mathrm{Homeo}}(M, \partial M)$ as a geometric realisation, a sub-basis for its topology is given by all subsets $C \subset \widetilde{\mathrm{Homeo}}(M, \partial M)$ such that the preimage of C under the quotient map

$$q\colon \sqcup_p \widetilde{\mathrm{Homeo}}(M, \partial M)_p \times \Delta^p \to \widetilde{\mathrm{Homeo}}(M, \partial M)$$

is a sub-basis element. A sub-basis element of $\sqcup_p \widetilde{\mathrm{Homeo}}(M, \partial M)_p \times \Delta^p$ consists of a union of sub-basis elements for the connected components, i.e. sets of the form $\sqcup_p (C_{M \times \Delta^p}(K, U) \times V)$ where $V \subset \Delta^p$ ranges over all sub-basis elements for Δ^p.

The preimage of such a set C under $\widetilde{\iota}$ is given by the intersection

$$q^{-1}(C) \cap \left(\left(\widetilde{\mathrm{Homeo}}(M, \partial M)_0 \times \Delta^0 \right) = \mathrm{Homeo}(M, \partial M) \right),$$

and by the definition this is a sub-basis element of $\mathrm{Homeo}(M, \partial M)$, completing the proof.

The proofs for (ii) proceeds almost identically, with the Whitney topology replacing the compact-open topology. The proof for (iii) is also similar, but one needs to use the fact that the inclusion $\mathrm{Diff}(M, \partial M) \to \mathrm{Homeo}(M, \partial M)$ is continuous. □

Lemma 9.1.4. *Let M be a smooth manifold. Then the following square commutes.*

$$\begin{array}{ccc} \mathrm{Diff}(M, \partial M) & \longrightarrow & \mathrm{Homeo}(M, \partial M) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Diff}}(M, \partial M) & \longrightarrow & \widetilde{\mathrm{Homeo}}(M, \partial M) \end{array} \tag{9.1.1}$$

Proof. This is clear from the definitions. □

Remark 9.1.5. Another approach (that might make Lemma 9.1.3 arise more naturally) is to also use simplicial spaces to define Homeo(−) and Diff(−). One could take the Definition 9.1.1 and additionally require that every simplex commutes with the projection to the standard simplex, i.e. turn all of the simplices into higher-order isotopies. One can show that the geometric resolution of such a simplicial space is homotopy equivalent to Homeo(−) or Diff(−), but now the first two maps in Lemma 9.1.3 can be induced by maps of simplicial spaces. We will not pursue this viewpoint further.

§ 9.2 | Pseudo-mapping class groups of 3-manifolds

We now restrict to considering 3-manifolds. The following result is basic.

Lemma 9.2.1. *Let Y be a smooth, compact 3-manifold. Then Equation (9.1.1) induces the following diagram on π_0.*

$$\begin{array}{ccc} \pi_0 \operatorname{Diff}(Y, \partial Y) & \xrightarrow{\cong} & \pi_0 \operatorname{Homeo}(Y, \partial Y) \\ \downarrow & & \downarrow \\ \pi_0 \widetilde{\operatorname{Diff}}(Y, \partial Y) & \twoheadrightarrow & \pi_0 \widetilde{\operatorname{Homeo}}(Y, \partial Y) \end{array}$$

Proof. The top horizontal map is an isomorphism due to Cerf [Cer59]. The bottom horizontal map being surjective is equivalent to every homeomorphism $Y \to Y$ being pseudo-isotopic to a diffeomorphism, which is true, since the work of Bing, Cairns and Moise [Cai40, Moi52, Bin54] says that every homeomorphism $Y \to Y$ is isotopic to a diffeomorphism. The two vertical arrows are clearly surjective by the definitions. □

An obvious question to ask is: which of the maps are injective? It turns out that, in many cases, they all are, due to the following deep theorem.

Theorem 9.2.2. *Let Y be a prime, compact 3-manifold, and let $f: Y \to Y$ and $g: Y \to Y$ be self-homeomorphisms. Then f is homotopic to g if and only if f is isotopic to g.*

We should say a few words about the citation for the above theorem, which the author did not find stated in the literature explicitly. Kwasik-Schultz and Hong-McCullough [KS89, HM13] collected references for certain classes of manifolds for which the "homotopy implies isotopy" theorem holds. We list these now.

(i) Haken manifolds, due to [Wal68].

(ii) Elliptic 3-manifolds, collectively due to Asano, Birman, Boileau, Bonahon, Cappell, Hodgson, Otal, Rubinstein and Shaneson. [Asa78, Rub79, Bon83, RB84, HR85, BO86].

(iii) Non-Haken Seifert fibred manifolds with infinite fundamental group, due to Scott and Boileau-Otal [Sco83, BO86, BO91].

(iv) Hyperbolic manifolds, due to Gabai, Meyerhoff and Thurston [Gab01, GMT03].

Proof of Theorem 9.2.2. Suppose Y is Haken. Then we know that Y satisfies the homotopy implies isotopy property by (i) above. Now assume Y is non-Haken but irreducible. If $\pi_1(Y)$ is finite, then geometrisation implies that Y is an elliptic 3-manifold, and hence we know it satisfies the theorem by (ii) above. Otherwise, $\pi_1(Y)$ is infinite, and then geometrisation implies that either Y is hyperbolic or it is Seifert fibred. By (iii) and (iv) above, both of these cases satisfy the theorem. □

Proposition 9.2.3. *Let Y be a prime, compact 3-manifold. Then the diagram in Lemma 9.2.1 becomes the following.*

$$\begin{array}{ccc} \pi_0 \operatorname{Diff}(Y, \partial Y) & \xrightarrow{\cong} & \pi_0 \operatorname{Homeo}(Y, \partial Y) \\ \downarrow \cong & & \downarrow \cong \\ \pi_0 \widetilde{\operatorname{Diff}}(Y, \partial Y) & \xrightarrow{\cong} & \pi_0 \widetilde{\operatorname{Homeo}}(Y, \partial Y) \end{array}$$

Proof. By Lemma 9.2.1, it suffices to show that the rightmost vertical map is injective. Then the top horizontal and the rightmost vertical maps are isomorphisms, and hence, since the remaining two maps are already known to be surjections, commutativity implies that the remaining maps are isomorphisms. The rightmost vertical map in Lemma 9.2.1 being injective is equivalent to the statement: if $f: Y \to Y$ is pseudo-isotopic to the identity, then f is isotopic to the identity. Since pseudo-isotopy implies homotopy (trivially), this means that pseudo-isotopy implies isotopy (by Theorem 9.2.2) and hence the rightmost vertical map is injective. □

The condition that Y be prime in Proposition 9.2.3 and Theorem 9.2.2 is, in general, necessary. In other words, there exist examples of closed 3-manifolds where pseudo-isotopy does not imply isotopy, which we will now introduce.

Definition 9.2.4. Let Γ be a finite group. We say that Γ is *metacyclic* if it is non-abelian but has finite cyclic Sylow-2 subgroup.

Metacyclic groups are one of the types of finite groups which act on S^3 freely, and hence give rise to elliptic manifolds (see [Thu97, Chapter 4.4])[2]. We call the class of 3-manifolds which arise as quotients of S^3 by metacyclic groups *metacyclic prism manifolds*.

We now describe a certain diffeomorphism of reducible 3-manifolds.

Definition 9.2.5. Let Y be compact, smooth 3-manifold such that $Y \cong Y_1 \# Y_2$ and let $S \subset Y$ denote the connected-sum S^2 for the given decomposition. Then we can write

$$Y \cong (\overline{Y_1 \setminus D^3}) \cup (S^2 \times I) \cup (\overline{Y_2 \setminus D^3}).$$

[2] An alternative description of metacyclic groups is as extensions of dihedral groups by certain cyclic groups. This is what is used in [Thu97].

We define a diffeomorphism on the $S^2 \times I$ as

$$\begin{aligned} T_S \colon S^2 \times I &\to S^2 \times I \\ (x,t) &\mapsto (R_t(x), t), \end{aligned}$$

where R_t denotes the (positive) rotation map of S^2 around the (oriented) straight line from the south pole to the north pole by an angle of t. The map T_S then extends to the rest of Y as the identity map, and we call this extension the *Dehn twist about* S, which we will also denote by T_S.

We then have the following result, which uses a result of Hatcher [Hat03].

Theorem 9.2.6 ([FW86, KS96])**.** *Let Y_1 and Y_2 be metacyclic prism manifolds. Further, let $Y := Y_1 \# Y_2$, and let S be the connected-sum sphere. Then $T_S \colon Y \to Y$ is topologically pseudo-isotopic to the identity, but not topologically isotopic to the identity.*

Corollary 9.2.7. *Let $Y := Y_1 \# Y_2$ a connected-sum of metacyclic prism manifolds. Then the rightmost vertical map in Lemma 9.2.1 is not injective. Hence, at least one of the maps*

$$\pi_0 \operatorname{Diff}(Y) \to \pi_0 \widetilde{\operatorname{Diff}}(Y)$$

or

$$\pi_0 \widetilde{\operatorname{Diff}}(Y) \to \pi_0 \widetilde{\operatorname{Homeo}}(Y)$$

is not injective.

Proof. The corollary is immediate from Lemma 9.2.1 and Theorem 9.2.6. □

In light of the above, we have the following natural question.

Question 9.2.8. *Let $Y := Y_1 \# Y_2$ be a connected-sum of metacyclic prism manifolds and let S denote the connected-sum sphere. Is the Dehn twist $T_S \colon Y \to Y$ smoothly pseudo-isotopic to the identity?*

We do not answer the above question here, but we hope to investigate it in future work.

§ 9.3 | Non-smoothable loops of homeomorphisms

We finish this chapter by proving the theorem stated at the start of the chapter, that, for elliptic 3-manifolds, the inclusion induced map

$$\widetilde{\operatorname{Diff}}(Y) \to \widetilde{\operatorname{Homeo}}(Y)$$

is not 1-connected. Compare this with the previous section, where we discussed a certain family of reducible 3-manifolds such that the composition

$$\operatorname{Diff}(Y) \to \widetilde{\operatorname{Diff}}(Y) \to \widetilde{\operatorname{Homeo}}(Y)$$

is not 0-connected. We also explained that for prime 3-manifolds, the map Section 9.3 induces an isomorphism on π_0.

Lemma 9.3.1. *Let M be a compact, smooth manifold and let $f\colon M \times I \to M \times I$ be a homeomorphism restricting to the identity on the boundary. Then $[f] \in \pi_1 \widetilde{\mathrm{Homeo}}(M)$ is in the image of $\pi_1 \widetilde{\mathrm{Diff}}(M) \to \pi_1 \widetilde{\mathrm{Homeo}}(M)$ if and only if f is pseudo-isotopic to a diffeomorphism.*

Proof. If f is pseudo-isotopic to a diffeomorphism, then there is a homeomorphism $F\colon (M \times I) \times I \to (M \times I) \times I$ such that $F_{M \times I \times \{0\}} = f$ and $F_{M \times I \times \{1\}} =: f'$ is a diffeomorphism, i.e. f' is in the image of $\pi_1 \widetilde{\mathrm{Diff}}(M) \to \pi_1 \widetilde{\mathrm{Homeo}}(M)$. The homeomorphism F then gives a 2-simplex in $\widetilde{\mathrm{Homeo}}(M)$ connecting f and f', and hence $[f] = [f']$.

Now assume $[f]$ is in the image of $\pi_1 \widetilde{\mathrm{Diff}}(M) \to \pi_1 \widetilde{\mathrm{Homeo}}(M)$. This means there is a homotopy in $\widetilde{\mathrm{Homeo}}(M)$ from f to a diffeomorphism f', which we may assume lies in 2-simplices of $\widetilde{\mathrm{Homeo}}(M)$. In other words, there is a sequence of 2-simplices connecting f and f' in $\widetilde{\mathrm{Homeo}}(M)$, and by concatenating these simplices we produce a pseudo-isotopy from f to a f'. □

For Y an elliptic 3-manifold, we want to use Theorem 8.0.1 to construct a homeomorphism $Y \times I \to Y \times I$ which is not pseudo-isotopic to a diffeomorphism, and hence prove Theorem 9.0.1 (by Lemma 9.3.1). To do this, we need to know that Theorem 8.0.1 applies to manifolds $Y \times I$ where Y is an elliptic 3-manifold. We show this now.

Proposition 9.3.2. *Let Γ be a fundamental group of an elliptic 3-manifold and let ρ denote the Sylow 2-subgroup of Γ. Then $SK_1(\mathbb{Z}[\rho])$ is trivial.*

Proof. The class of fundamental groups of elliptic 3-manifolds are exactly the class of finite groups which act freely on S^3. This class is well-understood, and splits into five subclasses: (i) finite cyclic groups, (ii) metacyclic prism groups, (iii) tetrahedral groups, (iv) octahedral groups, and (v) icosahedral groups. We elaborate on and deal with these cases separately.

(i) The Sylow 2-subgroup of a finite cyclic group is always cyclic and hence $SK_1(\mathbb{Z}[\rho])$ is trivial in these cases.

(ii) Metacyclic prism groups are exactly those groups which are non-abelian but have cyclic Sylow 2-subgroups. Hence, $SK_1(\mathbb{Z}[\rho])$ is trivial again in these cases.

(iii) This class of groups corresponds to products of the binary tetrahedral group (order 24) with a cyclic group of order coprime to 6. The Sylow 2-subgroup of such a group is isomorphic to Q_8 where Q_8 denotes the quarternion group. By Oliver [Oli88, Example 14.4] $SK_1(\mathbb{Z}[Q_8])_{(2)}$ is trivial. The Sylow p-subgroups of Q_8 are clearly trivial for $p \neq 2$, and hence by Oliver [Oli88, Theorem 14.2], $SK_1(\mathbb{Z}[Q_8])_{(p)}$ is trivial for all p. Hence, $SK_1(\mathbb{Z}[\rho])$ is trivial.

(iv) This class of groups corresponds to products of the binary octahedral group (order 48) with a cyclic group of order coprime to 6. The Sylow 2-subgroup of such a group is isomorphic Q_{16} where Q_{16} denotes the generalised quarternion group. The same analysis (and references) as in (iii) yields that $SK_1(\mathbb{Z}[\rho])$ is trivial.

(v) This class of groups corresponds to products of the binary icosahedral group (order 120) with a cyclic group of order coprime to 30. Like in the tetrahedral case, the Sylow 2-subgroup of such a group is isomorphic to Q_8, and hence $SK_1(\mathbb{Z}[\rho])$ is trivial.

This concludes the proof. □

Proof of Theorem 9.0.1. The proof is an application of Theorem 8.0.1 and Proposition 8.4.1. We will use these to construct a homeomorphism $Y \times I \to Y \times I$, restricting to the identity on the boundary, which is not pseudo-isotopic to any diffeomorphism. Such a homeomorphism represents an element in $\pi_1 \widetilde{\text{Homeo}}(Y)$, and this element is in the image of $\pi_1 \widetilde{\text{Diff}}(Y) \to \pi_1 \widetilde{\text{Homeo}}(Y)$ if and only if the homeomorphism is pseudo-isotopic to a diffeomorphism.

By Proposition 8.4.1 and Proposition 9.3.2, Theorem 8.0.1 applies to all manifolds $Y \times I$ where Y is an elliptic 3-manifold. By our assumption, $H_1(Y; \mathbb{Z}/2)$ is non-trivial, and hence $H^3(Y \times I, \partial; \mathbb{Z}/2)$ is non-trivial. Applying Theorem 8.0.1 then produces the required non-smoothable homeomorphism, completing the proof. □

Proposition 9.3.3. *The classes of elliptic* 3*-manifolds that Theorem 9.0.1 applies to are exactly the classes given in Remark 9.0.3 i.e. lens spaces with even-order fundamental group, metacyclic prism manifolds and octahedral manifolds.*

Proof. If Y is a lens space with odd-order fundamental group, tetrahedral or icosahedral then $H_1(Y; \mathbb{Z}/2) = 0$. For odd-order fundamental group lens spaces, this is clear. For tetrahedral manifolds and icosahedral manifolds, this is because the abelianisation of their fundamental group is always an odd-order cyclic group.

If Y is a lens space with even-order fundamental group, a metacyclic prism manifold, or octahedral, then $H_1(Y; \mathbb{Z}/2)$ is non-trivial. For even-order fundamnetal group lens spaces, this is clear. For metacyclic prism manifolds and octahedral manifolds, this is because the abelianisation of their fundamental group is always an even-order cyclic group.

The above two paragraphs cover all cases of elliptic 3-manifolds, hence we are done. □

Remark 9.3.4. If $Y \cong S^3$, then we know that the map $\pi_1 \widetilde{\text{Diff}}(Y) \to \pi_1 \widetilde{\text{Homeo}}(Y)$ is surjective. This can be deduced from [OP23, Corollary D], which shows that every self-homeomorphism of $S^3 \times I$ restricting to the identity on the boundary is isotopic, relative to the boundary, to the identity or the Dehn twist. Both of these are smooth, and hence there can be no non-smoothable loops of homeomorphisms of S^3.

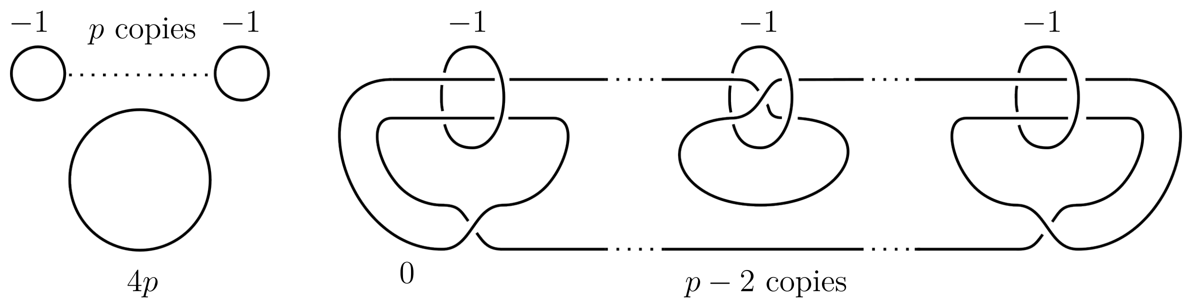


Figure 9.1: On the left: a surgery diagram for $Y\#(\#^p S^3)$. On the right: the surgery diagram after performing the handle slides described in the proof of Lemma 9.4.1.

§ 9.4 | Absolutely pseudo-smoothable homeomorphisms

We finish this chapter by proving our stated application to 4-manifolds, Theorem 9.0.4. First, we prove a lemma which gives specific fillings of certain elliptic 3-manifolds.

Lemma 9.4.1. *Let $Y := L(4p, 1)$ be a lens space. Then there exists a smooth, compact, oriented 4-manifold $X = X(4p)$ such that $\partial X \cong Y$ and such that the inclusion induced map on homology*

$$\mathbb{Z}/2 \cong H_1(Y; \mathbb{Z}/2) \to H_1(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is an isomorphism.

Proof. We build the filling using Kirby calculus. A surgery diagram for Y is given by one $4p$-framed unknot and p unlinked -1-framed unknots. Sliding the $4p$-framed unknot over each of the other components twice each produces the diagram shown on the right in Figure 9.1. A filling $X(4p)$ is then made by surgery on the 0-framed component of the link trace. One can compute that $\pi_1(X(4p)) \cong \mathbb{Z}/2$ and that the inclusion induced map on fundamental groups sends the generator of $\pi_1(Y)$ to the generator of $\pi_1(X)$. □

Proof of Theorem 9.0.4. Let $X := X(4p)$ be as in the proof of Lemma 9.4.1. One can verify that $\pi_1(X) \cong \mathbb{Z}/2$, and so we can apply Theorem 8.0.1 to produce a homeomorphism $\varphi\colon X \to X$ which has $\mathrm{cs}(\varphi) \neq 0 \in H^3(X, \partial X; \mathbb{Z}/2)$.

Similarly, there exists a homeomorphism $f\colon \partial X \times I \to \partial X \times I$ with $\mathrm{cs}(f) \neq 0 \in H^3(\partial X \times I, \partial; \mathbb{Z}/2)$. Let $f'\colon X \to X$ denote the extension of this homeomorphism by the identity map (here we have viewed $\partial X \times I$ as a collar of the boundary of X). By Proposition 5.4.2 and Lemma 9.4.1 it follows that

$$\mathrm{cs}(f' \circ \varphi) = \mathrm{cs}(f') + \mathrm{cs}(\varphi) = 0 \in H^3(X, \partial X; \mathbb{Z}/2)$$

and hence by Proposition 5.4.5 $f' \circ \varphi$ is stably pseudo-smoothable. In other words, there exists a $k \geq 0$ such that

$$(f' \circ \varphi)\#\,\mathrm{Id}_{\#^k(S^2\times S^2)}\colon X\#(\#^k S^2 \times S^2) \to X\#(\#^k S^2 \times S^2)$$

is pseudo-isotopic to a diffeomorphism.

The above means that $\varphi \# \operatorname{Id}_{\#^k(S^2\times S^2)}$ is absolutely pseudo-smoothable, since it is absolutely pseudo-isotopic to $(f' \circ \varphi) \# \operatorname{Id}_{\#^k(S^2\times S^2)}$, which is pseudo-smoothable. But, by Proposition 5.4.4, $\varphi \# \operatorname{Id}_{\#^k(S^2\times S^2)}$ is not pseudo-smoothable, and hence this completes the proof. □

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